

W*-superrigidity for wreath products with groups having positive first ℓ^2 -Betti number

by Mihaita Berbec¹

Abstract

In [BV12] we have proven that, for all hyperbolic groups and for all non-trivial free products Γ , the left-right wreath product group $\mathcal{G} := (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ is W*-superrigid, in the sense that its group von Neumann algebra $L\mathcal{G}$ completely remembers the group \mathcal{G} . In this paper, we extend this result to other classes of countable groups. More precisely, we prove that for weakly amenable groups Γ having positive first ℓ^2 -Betti number, the same wreath product group \mathcal{G} is W*-superrigid.

1 Introduction and main result

To any countable discrete group Γ we can associate the group von Neumann algebra $L\Gamma$ generated by the image of the left regular representation of Γ on the Hilbert space $\ell^2(\Gamma)$. This construction goes back to Murray and von Neumann [MvN43] and it provides a very rich source of examples of von Neumann algebras. The most interesting case is when $L\Gamma$ has trivial center, corresponding to Γ having infinite conjugacy classes (i.c.c.), i.e. Γ is infinite and all of its conjugacy classes, except for the trivial one, are infinite. In this case, $L\Gamma$ is a II_1 factor, this is an infinite dimensional von Neumann algebra that has trivial center and admits a positive trace.

One of the main problems in the theory of von Neumann algebras is to classify the group factors $L\Gamma$ in terms of the group Γ . More precisely, we are interested in answering the following question: does the group factor $L\Gamma$ *remember* the group Γ ? This natural question leads to two important concepts: *softness*, this is when $L\Gamma$ does not remember the group Γ , and *rigidity*, when $L\Gamma$ completely remembers the group Γ . In the first case, there is a long list of examples of groups that are soft. The celebrated theorem of Connes [Co76] says that all group II_1 factors arising from i.c.c. amenable groups are isomorphic to the hyperfinite II_1 factor. This shows that amenable groups manifest a remarkable softness: all the algebraic properties of the group, except their amenability, are lost when we pass to the group von Neumann algebra. In [Dy93], Dykema proved that for $\Gamma_1, \dots, \Gamma_n$ infinite amenable, $n \geq 2$, the group von Neumann algebra of their free product $L(\Gamma_1 * \dots * \Gamma_n)$ is isomorphic to the free group factor $L\mathbb{F}_n$. Ioana and Bowen obtained the first results saying that plain wreath products tend to be soft, namely all $L(\mathbb{F}_n \wr \mathbb{Z})$, for $n \geq 2$, are isomorphic [Io06] and all $L(H \wr \mathbb{F}_2)$, for H non-trivial abelian, are isomorphic [Bo09a],[Bo09b]. Moreover, in [IPV10], Ioana, Popa and Vaes proved that there exist infinitely many non-isomorphic countable groups Λ such that $L\Lambda$ is isomorphic to $L(\mathbb{Z}/2\mathbb{Z} \wr \text{PSL}(n, \mathbb{Z}))$, $n \geq 2$.

On the other hand, it is a famous open problem whether the free group factors $L\mathbb{F}_n$, with $n \geq 2$, are isomorphic or not. Another big open problem is *Connes' rigidity conjecture*. In [Co80a],[Co80b], Connes asked whether two i.c.c. property (T) groups Γ and Λ , with isomorphic group von Neumann algebras $L\Gamma \cong L\Lambda$, must necessarily be isomorphic. This conjecture remains wide open, even for classical groups like $\text{SL}(n, \mathbb{Z})$, with $n \geq 3$. Remark, however, that

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by [CH89], whenever Γ and Λ are lattices in $\mathrm{Sp}(n, 1)$, respectively $\mathrm{Sp}(m, 1)$, the isomorphism $L\Gamma \cong L\Lambda$ implies that $n = m$.

In [IPV10], Ioana, Popa and Vaes established the first W^* -superrigidity theorem for group von Neumann algebras: for a large class of generalized wreath product groups $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes \Gamma$, it was shown that if $LG \cong L\Lambda$, for an arbitrary countable group Λ , then G must be isomorphic with Λ . Such a group G is said to be W^* -superrigid (see Definition 1.1), and in this case the group von Neumann algebra LG completely remembers G .

Following the same strategy as in [IPV10], we have proven in [BV12] that the more natural *left-right wreath product* groups $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ are W^* -superrigid, where the direct product $\Gamma \times \Gamma$ acts on Γ by left-right multiplication, and where Γ is either the free group \mathbb{F}_n , with $n \geq 2$, or any i.c.c. hyperbolic group, or any non-trivial free product $\Gamma_1 * \Gamma_2$.

In this paper, we enlarge the class of groups covered by [BV12], proving that the left-right wreath product $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ is W^* -superrigid whenever Γ belongs to a certain class of groups with positive first ℓ^2 -Betti number or is a certain non-trivial amalgamated free product or HNN extension. For the precise statement we refer to Theorem 1.2.

Recall from [CH89] that a countable group Γ is said to be *weakly amenable* if it admits a sequence of finitely supported functions $\varphi_n : \Gamma \rightarrow \mathbb{C}$ tending to 1 pointwise and satisfying $\sup_n \|\varphi_n\|_{\mathrm{cb}} < \infty$, where $\|\varphi\|_{\mathrm{cb}}$ denotes the Herz-Schur norm of φ (i.e. the cb-norm of the linear map $L\Gamma \ni u_g \mapsto \varphi(g)u_g \in L\Gamma$).

If Γ is a countable group and $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K}_{\mathbb{R}})$ is an orthogonal representation of Γ on a real Hilbert space $\mathcal{K}_{\mathbb{R}}$, then a *1-cocycle* c into π is a map $c : \Gamma \rightarrow \mathcal{K}_{\mathbb{R}}$ satisfying the following cocycle relation:

$$c(gh) = c(g) + \pi(g)c(h), \text{ for all } g, h \in \Gamma.$$

A subgroup $\Sigma < \Gamma$ is called *malnormal* if $\Sigma \cap g\Sigma g^{-1} = \{1\}$, for all $g \in \Gamma \setminus \Sigma$. A subgroup $\Sigma < \Gamma$ is said to be *relatively malnormal* if there exists an infinite index subgroup $\Lambda < \Gamma$ such that $\Sigma \cap g\Sigma g^{-1}$ is finite, for all $g \in \Gamma \setminus \Lambda$. If $\{\Sigma_i\}_{i \in I}$ is a family of subgroups of Γ , then we say that $\{\Sigma_i\}_{i \in I}$ is *malnormal* in Γ if $g\Sigma_i g^{-1} \cap \Sigma_j = \{1\}$, unless $i = j$ and $g \in \Sigma_i$.

If Γ is a countable group, $\Sigma < \Gamma$ is a subgroup and $\theta : \Sigma \rightarrow \Gamma$ is an injective group homomorphism, then the HNN extension $\mathrm{HNN}(\Gamma, \Sigma, \theta)$ is the group generated by a copy of Γ and an extra generator t , called *stable letter*, subject to relations $tgt^{-1} = \theta(g)$, for all $g \in \Sigma$. We say that $\mathrm{HNN}(\Gamma, \Sigma, \theta)$ is *non-degenerate* if $\Sigma \neq \Gamma \neq \theta(\Sigma)$. Note that, in this case, $\mathrm{HNN}(\Gamma, \Sigma, \theta)$ contains a copy of the free group on two generators, hence it is non-amenable. In the same spirit, we say that an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is *non-degenerate* if $[\Gamma_1 : \Sigma] \geq 2$ and $[\Gamma_2 : \Sigma] \geq 3$, and this is sufficient to witness the non-amenability of Γ .

The group von Neumann algebra of an HNN extension $\mathrm{HNN}(\Gamma, \Sigma, \theta)$ is precisely the HNN extension of von Neumann algebras $\mathrm{HNN}(L\Gamma, L\Sigma, \Theta)$, associated to the triple $(L\Gamma, L\Sigma, \Theta)$, where Θ is the trace-preserving embedding $L\Sigma \rightarrow L\Gamma$ induced by θ . For more details about HNN extensions of von Neumann algebras see [Ue05] and [FV10, Section 3].

Definition 1.1. A countable group \mathcal{G} is said to be W^* -superrigid if for any countable group Λ such that $\pi : L\Lambda \rightarrow L\mathcal{G}$ is a $*$ -isomorphism, there exist a group isomorphism $\delta : \Lambda \rightarrow \mathcal{G}$, a character $\omega : \Lambda \rightarrow \mathbb{T}$ and a unitary $w \in \mathcal{U}(L\mathcal{G})$ such that

$$\pi(v_s) = \omega(s) w u_{\delta(s)} w^* \text{ for all } s \in \Lambda,$$

where $(v_s)_{s \in \Lambda}$ and $(u_g)_{g \in \mathcal{G}}$ denote the canonical generating unitaries of $L\Lambda$, respectively $L\mathcal{G}$.

We are now ready to state the main theorem of the paper.

Theorem 1.2. *Assume that Γ is one of the following countable groups:*

1. *a non-degenerate amalgamated free product $\Gamma_1 *_\Sigma \Gamma_2$, with Σ malnormal in Γ_1 ;*
2. *a non-degenerate HNN extension $\text{HNN}(\Gamma_0, \Sigma, \theta)$, with $\{\Sigma, \theta(\Sigma)\}$ malnormal in Γ_0 ;*
3. *an i.c.c. weakly amenable group with positive first ℓ^2 -Betti number that admits a bound on the order of its finite subgroups.*

Consider the action of $\Gamma \times \Gamma$ on Γ by left-right multiplication. Then the left-right wreath product group $\mathcal{G} = (\mathbb{Z}/n\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, with $n \in \{2, 3\}$, is W^ -superrigid in the sense of Definition 1.1.*

By [BV97], [PT07], a countable group Γ has positive first ℓ^2 -Betti number if and only if it is non-amenable and it admits an unbounded 1-cocycle into the left regular representation. Actually, throughout this paper we will only use this characterization of having positive first ℓ^2 -Betti number, without defining explicitly ℓ^2 -Betti numbers for countable groups. In [PT07, Section 3], there are given many examples of countable groups Γ with positive first ℓ^2 -Betti number, such as certain amalgamated free products, certain HNN extensions, hyperbolic triangle groups, limit groups of Sela, etc. Moreover, [PT07, Theorem 3.2] provides a very useful formula for estimating from below the first ℓ^2 -Betti number of a group defined by (a finite number of) generators and relations.

It is known that all Coxeter groups are weakly amenable ([Ja98],[Val93]). Using [PT07, Theorem 3.2] one can construct Coxeter groups with positive first ℓ^2 -Betti number and which are not hyperbolic (for details, see [KN11]). Remark that such groups give examples of groups that satisfy the third set of assumptions in Theorem 1.2 and that are not covered by [BV12, Theorem B].

Structure of the proof

Let Γ be a countable group satisfying one set of assumptions of Theorem 1.2. Denote $H := \mathbb{Z}/n\mathbb{Z}$, with $n \in \{2, 3\}$, and consider the wreath product $\mathcal{G} := H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, where $\Gamma \times \Gamma$ acts on Γ by left-right multiplication. Put $M := L\mathcal{G}$ and assume that Λ is an arbitrary countable group such that $M \cong L\Lambda$. We want to prove that \mathcal{G} is W^* -superrigid in the sense of Definition 1.1.

The proof follows exactly the same strategy as in [IPV10] and [BV12] and uses many results of these two papers. To describe more precisely this strategy, consider the comultiplication $\Delta : L\Lambda \rightarrow L\Lambda \overline{\otimes} L\Lambda$, defined by $\Delta(v_s) = v_s \otimes v_s$, for all $s \in \Lambda$, associated to the group von Neumann algebra decomposition $M \cong L\Lambda$. We write $A := LH^{(\Gamma)}$ and $G := \Gamma \times \Gamma$, so that $M = A \rtimes G$.

Under these assumptions, we prove that the following three statements are true:

$$\begin{aligned}
\Delta(A) &\prec^f A \overline{\otimes} A, \\
\Delta(A)' \cap M \overline{\otimes} M &\prec^f A \overline{\otimes} A, \\
\Omega \Delta(LG) \Omega^* &\subset LG \overline{\otimes} LG,
\end{aligned} \tag{1.1}$$

for some unitary $\Omega \in \mathcal{U}(M \overline{\otimes} M)$. Here, the notation " \prec^f " refers to Popa's intertwining-by-bimodules that we introduce in Section 2.

Having these three facts established, we can literally repeat the proof of [BV12, Theorem 8.1], followed by the proof of [BV12, Theorem B], in the particular case $H_0 = H$. This exactly yields the conclusion of Theorem 1.2.

All these statements, as well as the final argument, are showed to be true in Section 6. The rest of the paper is organized as follows. In Section 2 we introduce several preliminary notions and prove a number of technical lemmas that we need for the proof of the main theorem. In Section 3 and Section 4 we introduce the *malleable deformations*, in the sense of Popa, that we can define on our wreath product group von Neumann algebra, the *tensor length deformation* coming from the wreath product structure and the *Gaussian deformation* coming from the 1-cocycle into the left regular representation. In Section 5 we establish results that allow us to have good control on the normalizer of relatively amenable subalgebras.

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2 Preliminaries

Popa's intertwining-by-bimodules

Let (M, τ) be a tracial von Neumann algebra. Suppose that p and q are non-zero projections in M and that $P \subset pMp$ and $Q \subset qMq$ are von Neumann subalgebras. We recall briefly Popa's *intertwining-by-bimodules* definition/theorem.

Definition 2.1. We write $P \prec_M Q$ if there exists a non-zero P - Q -bimodule $\mathcal{H} \subset pL^2(M)q$ which has finite right Q -dimension. We write $P \prec_M^f Q$ if $Pp' \prec Q$, for all non-zero projections $p' \in P' \cap pMp$. If no confusion is possible, we simply write $P \prec Q$ and $P \prec^f Q$.

Theorem 2.2 ([Po03, Theorem 2.1 and Corollary 2.3]). *Suppose that P is generated by a group of unitaries $\mathcal{G} \subset \mathcal{U}(P)$. The following statements are equivalent:*

- $P \prec_M Q$;
- *There exist a non-zero projection $q_0 \in M_n(\mathbb{C}) \otimes Q$, a non-zero partial isometry $v \in M_{1,n}(\mathbb{C}) \otimes pMq$ and a normal $*$ -homomorphism $\theta : P \rightarrow q_0(M_n(\mathbb{C}) \otimes Q)q_0$ such that*

$$av = v\theta(a), \text{ for all } a \in P;$$

- *There is no sequence of unitaries $(u_n)_n$ in \mathcal{G} such that*

$$\|E_Q(xu_ny)\|_2 \rightarrow 0, \text{ for all } x, y \in pMq.$$

The next lemma is essentially a variant of [Po01, Theorem A.1], but we give a complete proof.

Lemma 2.3. *Let M be a type II_1 factor and $A \subset M$ be a Cartan subalgebra. Let $B \subset M$ be an abelian subalgebra and $\mathcal{G} < \mathcal{N}_M(B)$ be a subgroup such that*

- $B' \cap M \prec A$,
- the normalizer of $B' \cap M$ in M is a factor (or equivalently, the adjoint action of \mathcal{G} on $Z(B' \cap M)$ is ergodic).

Then there exist a projection $p \in A$ and an element $v \in M_{1,n}(\mathbb{C}) \otimes Mp$ such that $vv^* = 1$, $v^*v = 1 \otimes p$ and $v^*(B' \cap M)v = M_n(\mathbb{C}) \otimes Ap$.

Proof. Since $B' \cap M \prec A$, the von Neumann algebra $B' \cap M$ has a type I direct summand. Since the adjoint action of \mathcal{G} on $Z(B' \cap M)$ is ergodic, we find an integer $n \geq 1$ such that $B' \cap M = M_n(\mathbb{C}) \otimes Z(B' \cap M)$. So, we may take a system of matrix units $(e_{ij})_{1 \leq i, j \leq n}$ in $B' \cap M$ with $e := e_{11}$ satisfying $e(B' \cap M)e = Z(B' \cap M)e$. By construction, $Z(B' \cap M)e$ is a maximal abelian subalgebra of eMe , whose normalizer is a factor.

Since $B' \cap M \prec A$, also $Z(B' \cap M)e \prec A$ and hence, by [Po01, Theorem A.1], there exist a projection $p \in A$ and $v_0 \in M_{n,1}(\mathbb{C}) \otimes Mp$ such that $v_0v_0^* = e$, $v_0^*v_0 = p$ and $v_0^*(B' \cap M)v_0 = Ap$.

Define $v \in M_{1,n}(\mathbb{C}) \otimes Mp$ by $v = \sum_{k=1}^n e_{1k} \otimes e_{1k}v_0$. Then one checks easily that $vv^* = 1$, $v^*v = 1 \otimes p$ and $v^*(B' \cap M)v = M_n(\mathbb{C}) \otimes Ap$.

□

Jones' basic construction

Let (M, τ) be a tracial von Neumann algebra and $Q \subset M$ be a von Neumann subalgebra. The *Jones' basic construction* for the inclusion $Q \subset M$ is defined as the von Neumann algebra $\langle M, e_Q \rangle$ generated by M and the orthogonal projection $e_Q : L^2(M) \rightarrow L^2(Q)$.

We list now the main properties of the basic construction. Denote by Me_QM the linear span of the set $\{xe_Qy \mid x, y \in M\}$.

If Q is a von Neumann subalgebra of a tracial von Neumann algebra (M, τ) , then the basic construction $\langle M, e_Q \rangle$ is a semifinite von Neumann algebra, with a faithful normal semifinite trace Tr , satisfying the following properties:

- $\langle M, e_Q \rangle$ equals the commutant of the right action of Q on $L^2(M)$ and the $*$ -subalgebra Me_QM is weakly dense in $\langle M, e_Q \rangle$;
- $\text{Tr}(xe_Qy) = \tau(xy)$, for all $x, y \in M$;
- $e_Qxe_Q = E_Q(x)e_Q = e_QE_Q(x)$, for all $x \in M$;
- the central support of e_Q in $\langle M, e_Q \rangle$ is 1;
- Me_QM is dense in $L^2(\langle M, e_Q \rangle)$ in $\|\cdot\|_{2, \text{Tr}}$ -norm.

Part of these properties characterize the basic construction, as in the following well-known result (see e.g. [SS08, Theorem 3.3.15]).

Theorem 2.4. *Let N be a semifinite von Neumann algebra with a faithful normal semifinite trace Tr and von Neumann subalgebras $Q \subset M \subset N$. Assume that $e \in N$ is a projection such that*

1. N is the weak closure of the $*$ -subalgebra MeM ;

2. $\text{Tr}(e) = 1$ and $\tau(x) := \text{Tr}(xe)$ defines a faithful normal trace τ on M ;
3. $eNe = Qe = eQ$;

Then there is a trace-preserving $*$ -isomorphism $\theta : \langle M, e_Q \rangle \rightarrow N$ with $\theta(x) = x$, for all $x \in M$, and $\theta(e_Q) = e$.

Lemma 2.5. *Let $\Gamma \curvearrowright (A, \tau)$ be a trace-preserving action of a countable group Γ on a tracial von Neumann algebra (A, τ) . Let $\Sigma < \Gamma$ be a subgroup and denote $M := A \rtimes \Gamma$ and $Q := A \rtimes \Sigma$. Then the basic construction $\langle M, e_Q \rangle$ is isomorphic to $N = (A \overline{\otimes} \ell^\infty(\Gamma/\Sigma)) \rtimes \Gamma$, where Γ acts diagonally on $A \overline{\otimes} \ell^\infty(\Gamma/\Sigma)$.*

Proof. Define the projection $e := 1 \otimes \delta_{e_\Sigma} \in N$ and notice that we can see Q and M as subalgebras of the semifinite von Neumann algebra $N = (A \overline{\otimes} \ell^\infty(\Gamma/\Sigma)) \rtimes \Gamma \cong (A \rtimes \Sigma) \overline{\otimes} B(\ell^2(\Gamma/\Sigma))$. One can easily check that $Q \subset M \subset N$ and e satisfy all assumptions of Lemma 2.4 and hence the basic construction $\langle M, e_Q \rangle$ is isomorphic to N . \square

Relative amenability

Definition 2.6 ([OP07, Section 2.2]). Let (M, τ) be a tracial von Neumann algebra, $p \in M$ be a non-zero projection and let $P \subset pMp$ and $Q \subset M$ be von Neumann subalgebras. We say that P is *amenable relative to Q* inside M if there exists a P -central positive functional on the basic construction $p\langle M, e_Q \rangle p$, whose restriction to pMp equals the trace τ .

Following [IPV10], we say that P is *strongly non-amenable relative to Q* inside M if, for all non-zero projections $q \in P' \cap pMp$, we have that Pq is non-amenable relative to Q inside M .

Definition 2.7. Let (M, τ) and (N, τ) be tracial von Neumann algebras. Let $P \subset M$ be a von Neumann subalgebra. An M - N -bimodule ${}_M \mathcal{H}_N$ is said to be *left P -amenable* if $B(\mathcal{H}) \cap (N^{\text{op}})'$ admits a P -central state whose restriction to M equals the trace τ .

For more details about relative amenability and about left amenability for bimodules, see [Si10] and [PV11]. The link between these two notions is spelt out in the following remark.

Remark 2.8. If (M, τ) is a tracial von Neumann algebra and $P \subset pMp$ and $Q \subset M$ are von Neumann subalgebras, then by definition, P is amenable relative to Q inside M if and only if the pMp - Q -bimodule $pL^2(M)$ is left P -amenable.

The following criterion for relative amenability is due to [OP07] (see also [PV11, Section 2.5]). Here we copy the formulation of [BV12, Lemma 2.10].

Lemma 2.9 ([OP07, Corollary 2.3]). *Let (M, τ) be a tracial von Neumann algebra and $P \subset pMp$ be a von Neumann subalgebra. Let \mathcal{H} be a pMp - M -bimodule. Assume that $(\xi_i)_{i \in I} \in \mathcal{H}$ is a net of vectors satisfying the following three conditions:*

- $\limsup_{i \in I} \|\xi_i\| > 0$;
- $\limsup_{i \in I} \|x\xi_i\| \leq \|x\|_2$, for all $x \in pMp$;
- $\lim_{i \in I} \|a\xi_i - \xi_i a\| = 0$, for all $a \in P$.

Then there exists a non-zero projection $q \in P' \cap pMp$ such that the qMq - M -bimodule $q\mathcal{H}$ is left Pq -amenable.

Lemma 2.10. *Let (M, τ) be a tracial von Neumann algebra and assume that $M \subset \widetilde{M}$, for some von Neumann algebra \widetilde{M} . Let $S \subset M$ be a subset and let Ω be a positive functional on \widetilde{M} such that the restriction of Ω to M is bounded by $c\tau$, for some constant $c > 0$. If Ω is S -central, then Ω is S'' -central.*

Proof. For all elements $y \in \widetilde{M}$ and $x \in M$, by the Cauchy-Schwarz inequality, we have that $|\Omega(yx)|^2 \leq \Omega(y^*y)\Omega(x^*x) \leq c\Omega(y^*y)\tau(x^*x) \leq c\|y\|^2 \cdot \|x\|_2^2$ and similarly $|\Omega(xy)|^2 \leq c\|y\|^2 \cdot \|x\|_2^2$.

Thus, the set $M_0 := \{x \in M \mid \Omega(xy) = \Omega(yx) \text{ for all } y \in \widetilde{M}\}$ is an L^2 -closed $*$ -subalgebra of M . Since S is contained in M_0 and M_0 is L^2 -closed, it follows that S'' is also contained in M_0 , and this exactly means that Ω is S'' -central. \square

Lemma 2.11. *Let $\sigma : \Gamma \curvearrowright (X, \mu)$ be a free p.m.p. action of a countable group Γ on a standard probability space (X, μ) . Denote $A := L^\infty(X, \mu)$ and let $p \in A$ be a non-zero projection. Let $\Sigma < \Gamma$ be a subgroup, $n \geq 1$ be an integer and denote $M := M_n(\mathbb{C}) \otimes p(A \rtimes \Gamma)p$ and $Q := M_n(\mathbb{C}) \otimes p(A \rtimes \Sigma)p$. Assume that $\mathcal{G} < \mathcal{U}(M)$ is a subgroup and $q \in \mathcal{G}' \cap M$ is a non-zero projection such that*

- \mathcal{G} normalizes $M_n(\mathbb{C}) \otimes Ap$,
- $(\mathcal{G}q)''$ is amenable relative to Q .

Denote by M_0 the von Neumann algebra generated by \mathcal{G} and $1 \otimes Ap$. Then there exists a non-zero projection $q_0 \in M'_0 \cap M$ such that M_0q_0 is amenable relative to Q .

Proof. Since $(\mathcal{G}q)''$ is amenable relative to Q , there exists a state Ω_1 on $q\langle M, e_Q \rangle q$ such that Ω_1 is $\mathcal{G}q$ -central and it restricts to the trace on qMq .

Denote $N := M_n(\mathbb{C}) \otimes (p \otimes 1)((A \overline{\otimes} \ell^\infty(\Gamma/\Sigma)) \rtimes \Gamma)(p \otimes 1)$, where $\sigma : \Gamma \curvearrowright A \overline{\otimes} \ell^\infty(\Gamma/\Sigma)$ is the diagonal action. By Lemma 2.5 it follows that N is isomorphic with the basic construction $\langle M, e_Q \rangle$, thus Ω_1 is a $\mathcal{G}q$ -central state on qNq whose restriction to qMq equals the trace.

Define a state Ω on N by the formula $\Omega(T) = \Omega_1(qTq)$, for all $T \in N$. Since q commutes with \mathcal{G} , it follows immediately that Ω is \mathcal{G} -central. Since Ω_1 restricts to the trace on qMq , we get that $\Omega|_M$ is bounded by a multiple of the trace.

Denote $D := M_n(\mathbb{C}) \overline{\otimes} Ap \overline{\otimes} \ell^\infty(\Gamma/\Sigma)$ and let $E_D : N \rightarrow D$ be the unique trace-preserving conditional expectation.

We claim that every unitary $v \in \mathcal{U}(M)$ that normalizes $M_n(\mathbb{C}) \otimes Ap$ also normalizes D inside N . Indeed, take $v \in \mathcal{N}_M(M_n(\mathbb{C}) \otimes Ap)$. For every $g \in \Gamma$, there exist a projection $p_g \in Ap$ and a unitary $v_g \in \mathcal{U}(M_n(\mathbb{C}) \otimes A\sigma_g(p_g))$ such that $\sum_{g \in \Gamma} p_g = 1$, $\sum_{g \in \Gamma} \sigma_g(p_g) = 1$ and $v(1 \otimes p_g) = v_g(1 \otimes u_g)$. If $x \in M_n(\mathbb{C}) \otimes Ap_g$, then it follows immediately that $v xv^* = v_g(\text{id} \otimes \sigma_g)(x)v_g^*$. Moreover, for every $x \in M_n(\mathbb{C}) \overline{\otimes} Ap_g \overline{\otimes} \ell^\infty(\Gamma/\Sigma)$, we get that $v xv^* = (v_g \otimes 1)(\text{id} \otimes \sigma_g)(x)(v_g^* \otimes 1)$, and since the right hand side belongs to D , our claim is proven.

If $v \in \mathcal{N}_M(M_n(\mathbb{C}) \otimes Ap)$, then $v \in \mathcal{N}_N(D)$ and since E_D is the unique trace-preserving conditional expectation from N onto D , it follows that $E_D(vTv^*) = vE_D(T)v^*$, for all $T \in N$. In particular, $E_D(vTv^*) = vE_D(T)v^*$, for all $v \in \mathcal{G}$ and $T \in N$.

Define a state $\widetilde{\Omega}$ on N by $\widetilde{\Omega}(T) = \Omega(E_D(T))$, for all $T \in N$. Since Ω is \mathcal{G} -central, the previous remark implies that $\widetilde{\Omega}$ is also \mathcal{G} -central. Since $E_D(M) \subset M_n(\mathbb{C}) \otimes Ap$, we have that $\widetilde{\Omega}|_M$

is bounded by a multiple of the trace. Notice that $\tilde{\Omega}$ is automatically $(1 \overline{\otimes} Ap)$ -central, since $1 \overline{\otimes} Ap$ commutes with D , and hence, by Lemma 2.10, it follows that $\tilde{\Omega}$ is an M_0 -central state on N whose restriction to M is bounded by a multiple of the trace. In particular, $\tilde{\Omega}$ is normal on M , and then, by [BV12, Lemma 2.9], there exists a non-zero projection $q_0 \in M'_0 \cap M$ such that $M_0 q_0$ is amenable relative to Q . \square

The next easy lemma is essentially contained in the proof of [Io12b, Theorem 7.1] and [DI12, Lemma 8.2], but we provide a short proof for completeness.

Lemma 2.12. *Let Γ be a non-degenerate amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ or a non-degenerate HNN extension $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta)$. Then Σ is not co-amenable in Γ .*

Proof. Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ and assume that Σ is co-amenable in Γ , i.e. there exists a Γ -invariant state φ on $\ell^\infty(\Gamma/\Sigma)$. Let \mathcal{F}_1 and \mathcal{F}_2 be the sets of words beginning with a letter in $\Gamma_1 \setminus \Sigma$, respectively $\Gamma_2 \setminus \Sigma$. Then $\Gamma = \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \Sigma$. Let $\pi : \Gamma \rightarrow \Gamma/\Sigma$ be the quotient map and define $\mathcal{G}_1 = \pi(\mathcal{F}_1)$ and $\mathcal{G}_2 = \pi(\mathcal{F}_2)$. Thus $\Gamma/\Sigma = \mathcal{G}_1 \sqcup \mathcal{G}_2 \sqcup \{e\Sigma\}$.

Since Γ is non-degenerate, we can take elements $g_1 \in \Gamma_1 \setminus \Sigma$ and $g_2, g_3 \in \Gamma_2 \setminus \Sigma$ such that $g_3^{-1}g_2 \notin \Sigma$. Then we have that $\Sigma \subset g_1\mathcal{F}_1$, $g_1\mathcal{F}_2 \subset \mathcal{F}_1$, $g_2\mathcal{F}_1 \subset \mathcal{F}_2$ and $g_3\mathcal{F}_1 \subset \mathcal{F}_2$, hence $e\Sigma \in g_1\mathcal{G}_1$, $g_1\mathcal{G}_2 \subset \mathcal{G}_1$, $g_2\mathcal{G}_1 \subset \mathcal{G}_2$ and $g_3\mathcal{G}_1 \subset \mathcal{G}_2$.

Since $g_3^{-1}g_2\mathcal{G}_1 \subset \mathcal{G}_2$, then $g_2\mathcal{G}_1 \cap g_3\mathcal{G}_1 = \emptyset$, and hence $g_2\mathcal{G}_1 \sqcup g_3\mathcal{G}_1 \subset \mathcal{G}_2$. For any subset $\mathcal{F} \subset \Gamma/\Sigma$, define $m(\mathcal{F}) := \varphi(\chi_{\mathcal{F}}) \in [0, 1]$. Then m is a finitely additive Γ -invariant probability measure on Γ/Σ . Since π is Γ -equivariant and m is a finitely additive Γ -invariant measure, it follows that $m(e\Sigma) \leq m(\mathcal{F}_1)$, $m(\mathcal{F}_2) \leq m(\mathcal{F}_1)$ and $2m(\mathcal{F}_1) \leq m(\mathcal{F}_2)$, hence $m(e\Sigma) = m(\mathcal{F}_1) = m(\mathcal{F}_2)$. But this implies that $m(\Gamma/\Sigma) = 0$, which is a contradiction.

Let now $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta) = \langle \Gamma_0, t \mid t s t^{-1} = \theta(s), \forall s \in \Sigma \rangle$ and assume that Σ is co-amenable in Γ , i.e. there exists a Γ -invariant state φ on $\ell^\infty(\Gamma/\Sigma)$. Denote by $\pi : \Gamma \rightarrow \Gamma/\Sigma$ the quotient map and, for any subset $\mathcal{F} \subset \Gamma/\Sigma$, define $m(\mathcal{F}) := \varphi(\chi_{\mathcal{F}})$. Then m is a finitely additive Γ -invariant probability measure on Γ/Σ .

Let $A \subset \Gamma$ and $B \subset \Gamma$ be sets of representatives of left cosets of Σ , respectively $\theta(\Sigma)$ in Γ , with $e \in A$ and $e \in B$. Since Γ is non-degenerate, we can take elements $a \in A \setminus \{e\}$ and $b \in B \setminus \{e\}$. Recall that every element $g \in \Gamma$ can be uniquely written as $g = g_n t^{\varepsilon_n} g_{n-1} \dots g_1 t^{\varepsilon_1} g_0$, where $g_0 \in A$, $g_i \in \Gamma_0$ and $\varepsilon_i \in \{-1, 1\}$, for all $i = 1, \dots, n$, such that $g_i \in A$ if $\varepsilon_i = -1$ and $g_i \in B$ if $\varepsilon_i = 1$.

Denote by S the set of all elements $g = g_n t^{\varepsilon_n} g_{n-1} \dots g_1 t^{\varepsilon_1} g_0 \in \Gamma$ such that $n \geq 1$ and $g_n \neq e$ and denote by U , respectively V , the set of all $g \in \Gamma$ such that $n \geq 1$, $g_n = e$ and $\varepsilon_n = -1$, respectively $\varepsilon_n = 1$. Then we have that $t^{-1}S \subset U$, $tS \subset V$, $aU \subset S$, $bV \subset S$ and $aU \cap bV = \emptyset$. Since π is Γ -equivariant and m is a finitely additive Γ -invariant measure, it follows that $m(\pi(S)) = m(\pi(V)) = m(\pi(U)) = 0$. Since $\Gamma/\Sigma = \pi(U) \cup \pi(V) \cup \pi(S) \cup \Gamma_0/\Sigma$, we get that $m(\Gamma_0/\Sigma) = 1$, which is a contradiction since $t(\Gamma_0/\Sigma) \cap \Gamma_0/\Sigma = \emptyset$. \square

Properties of the comultiplication

We recall now a few useful properties of the comultiplication that we shall use throughout the paper. Let M be a II_1 factor and assume that $M \cong \text{L}\Lambda$, for some countable group Λ . Define the comultiplication $\Delta : \text{L}\Lambda \rightarrow \text{L}\Lambda \overline{\otimes} \text{L}\Lambda$, associated to Λ , by

$$\Delta(v_s) = v_s \otimes v_s, \text{ for all } s \in \Lambda.$$

The next proposition is contained in [IPV10, proposition 7.2] and [BV12, Proposition 4.1].

Proposition 2.13. *Assume that \widetilde{M} is a tracial von Neumann algebra such that $M \subset \widetilde{M}$.*

1. *If $Q \subset M$ is a von Neumann subalgebra such that $\Delta(M) \prec M \overline{\otimes} Q$, then there exists a non-zero projection $q \in Q' \cap M$ such that $Qq \subset qMq$ has finite index.*
2. *If $Q \subset \widetilde{M}$ is a von Neumann subalgebra such that $\Delta(M)$ is amenable relative to $M \overline{\otimes} Q$ inside $M \overline{\otimes} \widetilde{M}$, then M is amenable relative to Q inside \widetilde{M} .*
3. *If $Q \subset M$ is a von Neumann subalgebra that has no amenable direct summand, then $\Delta(Q)$ is strongly non-amenable relative to $M \overline{\otimes} 1$.*
4. *If $Q \subset M$ is diffuse, then $\Delta(Q) \not\prec M \overline{\otimes} 1$ and $\Delta(Q) \not\prec 1 \overline{\otimes} M$.*
5. $\Delta(M)' \cap M \overline{\otimes} M = \mathbb{C}1$.

3 Tensor length deformation

Assume that G is a countable discrete group acting on a countable set I and let (A_0, τ) be any tracial von Neumann algebra. Consider the generalized Bernoulli action $G \curvearrowright A_0^I$. Denote by M the corresponding Bernoulli crossed product $M = A_0^I \rtimes G$.

In [Po03, Po04], Popa introduced his fundamental *malleable deformation* for Bernoulli crossed products and used it to prove the first W^* -rigidity theorems for property (T) groups. In [Po06b], Popa introduced spectral gap methods to prove W^* -rigidity theorems for direct products of non-amenable groups. These methods and results have been intensively generalized and used in many subsequent works. For more details about Popa's deformation/rigidity theory, we refer to the survey papers [Po06a], [Va10a], [Io12b].

In this paper, we use the following variant of Popa's malleable deformation for Bernoulli crossed products, due to Ioana [Io06]. Consider the free product $A_0 * \mathbb{L}\mathbb{Z}$, with respect to the natural traces. Denote $\widetilde{M} := (A_0 * \mathbb{L}\mathbb{Z})^I \rtimes G$ the corresponding generalized Bernoulli crossed product.

Consider the self-adjoint element $h \in \mathbb{L}\mathbb{Z}$, with spectrum $[-\pi, \pi]$, such that $\exp(ih)$ equals the canonical generating unitary of $\mathbb{L}\mathbb{Z}$ and denote by $(u_t)_{t \in \mathbb{R}}$ the one-parameter group of unitary elements in $\mathbb{L}\mathbb{Z}$ given by $u_t := \exp(iht)$, for all $t \in \mathbb{R}$.

Define a one-parameter group of automorphisms $\alpha_t \in \text{Aut}(\widetilde{M})$ by

$$\alpha_t(u_g) = u_g, \text{ for all } g \in G,$$

and

$$\alpha_t(\pi_i(x)) = \pi_i(u_t x u_t^*), \text{ for all } x \in A_0 * \mathbb{L}\mathbb{Z}, i \in I,$$

where $\pi_i : A_0 * \mathbb{L}\mathbb{Z} \rightarrow (A_0 * \mathbb{L}\mathbb{Z})^I$ puts an element of $A_0 * \mathbb{L}\mathbb{Z}$ in the i -th position in $(A_0 * \mathbb{L}\mathbb{Z})^I$.

We call $(\alpha_t)_{t \in \mathbb{R}} \in \text{Aut}(\widetilde{M})$ the *tensor length deformation* of the generalized Bernoulli crossed product $M = A_0^I \rtimes G$. If $Q \subset \widetilde{M}$ is a von Neumann subalgebra, then we say that $\alpha_t \rightarrow \text{id}$ uniformly on Q if

$$\sup_{b \in \mathcal{U}(Q)} \|\alpha_t(b) - b\|_2 \rightarrow 0, \text{ as } t \rightarrow 0.$$

Remark 3.1. If $Q \subset M$ is a von Neumann subalgebra such that $\alpha_t \rightarrow \text{id}$ uniformly on Q , then $\alpha_t \rightarrow \text{id}$ uniformly on $E_A(Q)$, where E_A is the conditional expectation from \widetilde{M} onto A .

Denote $\rho_t := |\tau(u_t)|^2$ and observe that $0 \leq \rho_t < 1$, for all $t \neq 0$, and that $\rho_t \rightarrow 1$, as $t \rightarrow 0$. Let $\mathcal{F} \subset I$ be a finite subset and denote by $\pi_{\mathcal{F}} : A_0^{\mathcal{F}} \rightarrow A_0^I$ the natural embedding. Whenever $a \in A_0^{\mathcal{F}}$ is the elementary tensor given by $a = \otimes_{i \in \mathcal{F}} a_i$, with $a_i \in A_0 \ominus \mathbb{C}1$, we have that $E_A(\alpha_t(\pi_{\mathcal{F}}(a))) = \rho_t^{|\mathcal{F}|} \pi_{\mathcal{F}}(a)$. Since we have that $\rho_t \rightarrow 1$, as $t \rightarrow 0$, a straightforward computation yields that $\sup_{b \in \mathcal{U}(Q)} \|\alpha_t(E_A(b)) - E_A(\alpha_t(b))\|_2 \rightarrow 0$, as $t \rightarrow 0$. Hence $\alpha_t \rightarrow \text{id}$ uniformly on $E_A(Q)$.

We recall from [BV12] the following *spectral gap rigidity* theorem.

Theorem 3.2 ([BV12, Theorem 3.1]). *Let $G \curvearrowright I$ be an action of a countable group on a countable set. Assume that (A_0, τ) and (N, τ) are arbitrary tracial von Neumann algebras. Consider, as above, the generalized Bernoulli crossed product $M = A_0^I \rtimes G$, with its tensor length deformation $(\alpha_t)_{t \in \mathbb{R}} \in \text{Aut}(\widetilde{M})$. Let $p \in N \overline{\otimes} M$ be a non-zero projection and $Q \subset p(N \overline{\otimes} M)p$ be a von Neumann subalgebra.*

Assume there exists an integer $\kappa > 0$ such that for every finite subset $\mathcal{F} \subset I$, with $|\mathcal{F}| \geq \kappa$,

$$Q \text{ is strongly non-amenable relative to } N \overline{\otimes} (A_0^I \rtimes \text{Stab } \mathcal{F}).$$

Then:

$$\text{id} \otimes \alpha_t \rightarrow \text{id} \text{ uniformly on } \mathcal{U}(Q' \cap p(N \overline{\otimes} M)p).$$

4 Cocycles and Gaussian deformation

Let Γ be a countable group and let $c : \Gamma \rightarrow \mathcal{K}_{\mathbb{R}}$ be a 1-cocycle into the orthogonal representation $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K}_{\mathbb{R}})$. The 1-cocycle c defines a one-parameter family $(\psi_t)_{t > 0}$ of positive definite functions on Γ by

$$\psi_t : \Gamma \ni g \mapsto \psi_t(g) := \exp(-t \|c(g)\|^2) \in \mathbb{R}.$$

If $\Gamma \curvearrowright (A, \tau)$ is a trace-preserving action of Γ on the tracial von Neumann algebra (A, τ) and $M := A \rtimes \Gamma$ is the corresponding crossed product, then to the family $(\psi_t)_{t > 0}$ corresponds a one-parameter family $(\varphi_t)_{t > 0}$ of unital completely positive normal trace-preserving maps on M , defined by

$$\varphi_t : M \ni bu_g \mapsto \varphi_t(bu_g) := \psi(g)bu_g = \exp(-t \|c(g)\|^2)bu_g \in M.$$

If $\mathcal{K}_{\mathbb{R}}$ is a real Hilbert space and $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K}_{\mathbb{R}})$ is an orthogonal representation, then we denote by \mathcal{K} the complexification of $\mathcal{K}_{\mathbb{R}}$ and by π the corresponding unitary representation on \mathcal{K} . To any unitary representation $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$ we associate the M - M -bimodule $K^{\pi} := \mathcal{K} \otimes L^2(M)$, where the left-right M -module action on K^{π} is given by

$$(bu_g) \cdot (\xi \otimes x) \cdot y = \pi(g)\xi \otimes (bu_g)xy,$$

for all $b \in A, g \in \Gamma, \xi \in \mathcal{K}$ and $x, y \in M$.

The unitary representation $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$ is said to be *mixing* if, for all $\xi, \eta \in \mathcal{K}$, we have that

$$\langle \pi(g)\xi, \eta \rangle \rightarrow 0, \text{ as } g \rightarrow \infty.$$

We define now the *malleable Gaussian deformation* ([Si10, Section 3]) on $M = A \rtimes \Gamma$, associated to the 1-cocycle $c : \Gamma \rightarrow \mathcal{K}_{\mathbb{R}}$ into the orthogonal representation $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K}_{\mathbb{R}})$.

Denote by $\sigma : \Gamma \curvearrowright (Y, \nu)$ the Gaussian action associated to the orthogonal representation π . Let $D := L^\infty(Y, \nu)$ and τ be the trace on D given by integration with respect to ν . Then σ yields a trace-preserving action $(\sigma_g)_{g \in \Gamma}$ of Γ on (D, τ) . For the purpose of our paper it is more convenient to see (D, τ) as the unique abelian tracial von Neumann algebra generated by unitaries $\omega(\xi)$, with $\xi \in \mathcal{K}_{\mathbb{R}}$, subject to the following relations:

$$\omega(\xi + \eta) = \omega(\xi)\omega(\eta), \text{ for all } \xi, \eta \in \mathcal{K}_{\mathbb{R}};$$

$$\omega(0) = 1, \omega(\xi)^* = \omega(-\xi), \text{ for all } \xi \in \mathcal{K}_{\mathbb{R}};$$

$$\tau(\omega(\xi)) = \exp(-\|\xi\|^2), \text{ for all } \xi \in \mathcal{K}_{\mathbb{R}}.$$

By construction, the Gaussian action of Γ on (D, τ) is given by

$$\sigma_g(\omega(\xi)) = \omega(\pi(g)\xi), \text{ for all } g \in \Gamma, \xi \in \mathcal{K}_{\mathbb{R}}.$$

We denote $\widetilde{M} := (D \overline{\otimes} A) \rtimes \Gamma$, where Γ acts diagonally on $D \overline{\otimes} A$, and we define a one-parameter group of automorphisms $(\beta_t)_{t \in \mathbb{R}} \in \text{Aut}(\widetilde{M})$ by

$$\beta_t(x) = x, \text{ for all } x \in D \overline{\otimes} A,$$

and

$$\beta_t(u_g) = (\omega(tc(g)) \otimes 1)u_g, \text{ for all } g \in \Gamma, t \in \mathbb{R}.$$

The automorphisms $(\beta_t)_{t \in \mathbb{R}} \in \text{Aut}(\widetilde{M})$ give a malleable deformation in the sense of Popa, i.e. $\beta_t \rightarrow \text{id}$ pointwise, as $t \rightarrow 0$, in the L^2 -norm on \widetilde{M} .

We record for later use the following two easy lemmas.

Lemma 4.1 ([Io11, Lemma 2.1]). *If $\beta_t \rightarrow \text{id}$ uniformly on the unit ball of pMp , for some non-zero projection $p \in M$, then the cocycle c must be bounded.*

Proof. Assume that $\beta_t \rightarrow \text{id}$ uniformly on the unit ball of pMp . Then $\beta_t \rightarrow \text{id}$ uniformly on the unit ball of Mz , where z is the central support of p in M . Therefore, we have that $\tau(\beta_t(u_g)u_g^*z) \rightarrow \tau(z)$, uniformly in $g \in \Gamma$. Since $E_M(\beta_t(u_g)) = \exp(-t^2 \|c(g)\|^2)u_g$ and since the conditional expectation E_M is trace-preserving, it follows that $\exp(-t^2 \|c(g)\|^2) \rightarrow 1$, uniformly in $g \in \Gamma$, and this implies that c is bounded. \square

Lemma 4.2. *Assume that M is a type II_1 factor and $M \cong L\Lambda$ for some countable group Λ . Define the comultiplication $\Delta : L\Lambda \rightarrow L\Lambda \overline{\otimes} L\Lambda$ by $\Delta(v_s) = v_s \otimes v_s$, for all $s \in \Lambda$.*

If $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the unit ball of $q\Delta(M)q$, for some non-zero projection $q \in M \overline{\otimes} M$, then the cocycle c must be bounded.

Proof. Let $q \in M \overline{\otimes} M$ be a non-zero projection and assume that $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the unit ball of $q\Delta(M)q$. Since $\{v_s\}_{s \in \Lambda}$ is a group of unitaries generating M , we have that $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on $\{q\Delta(v_s)q = q(v_s \otimes v_s)q \mid s \in \Lambda\}$. A combination of [Va10b, Lemma 3.3] and [Va10b, Lemma 3.4] yields a non-zero projection $q_1 \in M$ such that $q \leq 1 \otimes q_1$ and $\beta_t \rightarrow \text{id}$ uniformly on $\{q_1 v_s q_1 \mid v \in \Lambda\}$. Since the group of unitaries $\{v_s\}_{s \in \Lambda}$ generates M , applying again [Va10b, Proposition 3.4], it follows that $\beta_t \rightarrow \text{id}$ uniformly on the unit ball of $q_2 M q_2$, where q_2 is a projection in M satisfying $q_1 \leq q_2$. Since q_1 is non-zero, it follows that q_2 is also non-zero and then, by Lemma 4.1, the cocycle c must be bounded. \square

In [Pe09, Theorem 4.5] and [CP10, Theorem 2.5], using Peterson's techniques of unbounded derivations, it has been proven that whenever π is mixing and $\beta_t \rightarrow \text{id}$ uniformly on a von Neumann subalgebra $Q \subset M$ such that $Q \not\prec A$, then $\beta_t \rightarrow \text{id}$ uniformly on the normalizer of Q . An alternative proof of this result was given by Vaes, in [Va10b], using the Gaussian automorphisms $(\beta_t)_{t \in \mathbb{R}}$. The precise formulation of this result goes as follows.

Theorem 4.3 ([Va10b, Theorem 3.10]). *Assume that π is a mixing representation. Let $p \in M$ be a projection and $Q \subset pMp$ be a von Neumann subalgebra such that $Q \not\prec A$ and such that $\beta_t \rightarrow \text{id}$ uniformly on the unit ball of Qq , for some non-zero projection $q \in Q' \cap pMp$. Denote by P the normalizer of Q inside pMp . Then $\beta_t \rightarrow \text{id}$ uniformly on the unit ball of Pr , where r is the smallest central projection in $Z(P)$ satisfying $q \leq r$.*

5 Normalizers of (relatively) amenable subalgebras

Let Γ be a countable group and let $\Gamma \curvearrowright (A, \tau)$ be a trace-preserving action such that the crossed product $M := A \rtimes \Gamma$ is a type II₁ factor. Assume that $M \cong L\Lambda$, for some countable group Λ , and define the corresponding comultiplication $\Delta : L\Lambda \rightarrow L\Lambda \overline{\otimes} L\Lambda$.

The following two results are direct consequences of [PV11, Theorem 3.1] and [Va13, Theorems A and 4.1] and they are very similar to [BV12, Theorem 5.1].

Theorem 5.1. *Assume that Γ is non-amenable, weakly amenable and it admits an unbounded 1-cocycle $c : \Gamma \rightarrow \mathcal{K}_{\mathbb{R}}$ into a mixing orthogonal representation $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K}_{\mathbb{R}})$ that is weakly contained into the left regular representation of Γ .*

Let $Q \subset M \overline{\otimes} M$ be a von Neumann subalgebra such that $\Delta(M) \subset \mathcal{N}_{M \overline{\otimes} M}(Q)''$ and such that Q is amenable relative to $M \overline{\otimes} A$. Then $Q \prec^f M \overline{\otimes} A$.

Proof. Denote by \mathcal{K}^π the M - M -bimodule associated to π and by $(\varphi_t)_{t \geq 0}$ the group of unital normal completely positive maps associated to the 1-cocycle c .

Denote $P := \mathcal{N}_{M \overline{\otimes} M}(Q)''$. By Proposition 2.13.(5), we have that $\Delta(M)' \cap M \overline{\otimes} M = \mathbb{C}1$ and since, moreover $\Delta(M) \subset P$, it suffices to prove that $Q \prec M \overline{\otimes} A$.

Denote $\mathcal{M} := M \overline{\otimes} M$ and $\mathcal{A} := M \overline{\otimes} A$, so that $\mathcal{M} \cong \mathcal{A} \rtimes \Gamma$. Assume, by contradiction, that $Q \not\prec \mathcal{A} = M \overline{\otimes} A$. Then, by [PV11, Theorem 3.1], at least one of the following must be true:

1. The \mathcal{M} - \mathcal{M} -bimodule \mathcal{K}^π is left P -amenable;
2. There exist $t, \delta > 0$ such that $\|\varphi_t(a)\|_2 \geq \delta$, for all $a \in \mathcal{U}(Q)$.

Case 1. If ${}_{\mathcal{M}}\mathcal{K}^\pi_{\mathcal{M}}$ is left P -amenable, then ${}_{\mathcal{M}}\mathcal{K}^\pi_{\mathcal{M}}$ is left $\Delta(M)$ -amenable. Since π is weakly contained in the left regular representation, it follows that ${}_{\mathcal{M}}\mathcal{K}^\pi_{\mathcal{M}} \prec {}_{\mathcal{M}}(L^2(\mathcal{M}) \otimes_{\mathcal{A}} L^2(\mathcal{M}))_{\mathcal{M}}$, and therefore, by [PV11, Corollary 2.5], we get that

$${}_{\mathcal{M}}(L^2(\mathcal{M}) \otimes_{\mathcal{A}} L^2(\mathcal{M}))_{\mathcal{M}} \text{ is left } \Delta(M)\text{-amenable.}$$

By [PV11, Proposition 2.4] this further implies that ${}_{\mathcal{M}}L^2(\mathcal{M})_{\mathcal{A}}$ is left $\Delta(M)$ -amenable, i.e. $\Delta(M)$ is amenable relative to $\mathcal{A} = M \overline{\otimes} A$. Finally, by Proposition 2.13.(2), we get that M is amenable relative to A , which contradicts the non-amenableity of Γ .

Case 2. Assume that there exist $t, \delta > 0$ such that $\|\varphi_t(a)\|_2 \geq \delta$, for all $a \in \mathcal{U}(Q)$. Let $(\beta_t)_{t \in \mathbb{R}} \in \text{Aut}(\widetilde{M})$ be the Gaussian deformation on M , defined in Section 4.

Since π is mixing, by [Va10b, Proposition 3.9], there is a non-zero projection $p \in Z(P)$ such that

$$\text{id} \otimes \beta_t \rightarrow \text{id} \text{ uniformly on the unit ball of } Qp.$$

Now, since moreover $Q \not\prec \mathcal{A}$, it follows by Theorem 4.3 that

$$\text{id} \otimes \beta_t \rightarrow \text{id} \text{ uniformly on the unit ball of } Pq,$$

where $q \in Z(P)$ is the smallest projection such that $p \leq q$. In particular, q is non-zero and since $\Delta(M) \subset P$ we get that $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the unit ball of $\Delta(M)q$, but this contradicts Lemma 4.2. \square

Theorem 5.2. *Assume that Γ is an amalgamated free product $\Gamma_1 *_\Sigma \Gamma_2$ or an HNN extension $\text{HNN}(\Gamma_0, \Sigma, \theta)$ as in Theorem 1.2.(1), respectively 1.2.(2).*

Let $Q \subset M \overline{\otimes} M$ be a von Neumann subalgebra such that $\Delta(M) \subset \mathcal{N}_{M \overline{\otimes} M}(Q)''$ and such that Q is amenable relative to $M \overline{\otimes} A$. Then $Q \prec^f M \overline{\otimes} A$.

Proof. Denote $P := \mathcal{N}_{M \overline{\otimes} M}(Q)''$ and $\mathcal{A} := M \overline{\otimes} A$. Suppose first that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is non-degenerate and Σ is malnormal in Γ_1 and notice that Σ is relatively malnormal in Γ (indeed, Γ_2 has infinite index in Γ and $\Sigma \cap g\Sigma g^{-1}$ is finite, for all $g \in \Gamma \setminus \Gamma_2$). Remark also that we can write $M \overline{\otimes} M$ as an amalgamated free product

$$M \overline{\otimes} M = (\mathcal{A} \rtimes \Gamma_1) *_{\mathcal{A} \rtimes \Sigma} (\mathcal{A} \rtimes \Gamma_2).$$

By [Va13, Theorem A], at least one of the following statements is true:

- $Q \prec \mathcal{A} \rtimes \Sigma$;
- $P \prec \mathcal{A} \rtimes \Gamma_i$, for some $i = 1$ or 2 ;
- P is amenable relative to $\mathcal{A} \rtimes \Sigma$.

If $Q \prec \mathcal{A} \rtimes \Sigma$, then we get that $Q \prec \mathcal{A}$. Indeed, since Σ is relatively malnormal in Γ there is an infinite index subgroup $\Lambda < \Gamma$ such that $\Sigma \cap g\Sigma g^{-1}$ is finite, for all $g \in \Gamma \setminus \Lambda$. Assume, by contradiction, that $Q \not\prec \mathcal{A}$. Then, by [Va10b, Lemma 6.4], it follows that $P \prec \mathcal{A} \rtimes \Lambda$, and hence $\Delta(M) \prec \mathcal{A} \rtimes \Lambda$, which, by Proposition 2.13.(1), is not possible since Λ has infinite index in Γ . Thus we get that $Q \prec \mathcal{A} = M \overline{\otimes} A$. By Proposition 2.13.(5), we have that $\Delta(M)' \cap M \overline{\otimes} M = \mathbb{C}1$ and since moreover $\Delta(M) \subset P$, we get indeed that $Q \prec^f M \overline{\otimes} A$.

If $P \prec \mathcal{A} \rtimes \Gamma_i$, for some $i = 1$ or 2 , then $\Delta(M) \prec \mathcal{A} \rtimes \Gamma_i$, which contradicts Proposition 2.13.(1), since Γ_i has infinite index in Γ , for all $i = 1, 2$.

If P is amenable relative to $\mathcal{A} \rtimes \Sigma$, for some $i = 1$ or 2 , then $\Delta(M)$ is amenable relative to $\mathcal{A} \rtimes \Sigma$. By Proposition 2.13.(2) it follows that M is amenable relative to $A \rtimes \Sigma$, but this further implies that Σ is co-amenable in Γ , which contradicts Lemma 2.12.

Assume now that $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta) = \langle \Gamma_0, t \mid t\Sigma t^{-1} = \theta(\Sigma) \rangle$ is non-degenerate and $\{\Sigma, \theta(\Sigma)\}$ is malnormal in Γ_0 . By [KS70, Corollary 4, page 954] we have that Σ is malnormal in Γ , so in particular, Σ is relatively malnormal in Γ . Using the construction in [FV10, Section 3], we can write $M \overline{\otimes} M$ as an HNN extension $\text{HNN}(\mathcal{A} \rtimes \Gamma_0, \mathcal{A} \rtimes \Sigma, \Theta)$, and hence, by [Va13, Theorem 4.1], at least one of the following statements is true:

- $Q \prec \mathcal{A} \rtimes \Sigma$;

- $P \prec \mathcal{A} \rtimes \Gamma_0$;
- P is amenable relative to $\mathcal{A} \rtimes \Sigma$.

The last two alternatives cannot hold, as in the previous case, thus we have $Q \prec \mathcal{A} \rtimes \Sigma$, which implies that $Q \prec^f M \overline{\otimes} A$, since $\Sigma < \Gamma$ is relatively malnormal. \square

The next result is an analogue of [Io12a, Corollary 2.12]. Since the first part of the proof goes exactly as in Ioana's proof, we will be rather brief, pointing out the arguments that are different.

Theorem 5.3. *Assume that Γ is non-amenable and it admits an unbounded 1-cocycle c into the left regular representation of Γ .*

Let $\Sigma < \Gamma$ be a subgroup and assume that the cocycle c is bounded on Σ . Denote $M_1 := A \rtimes \Sigma$ and let $Q \subset pMp$ be a von Neumann subalgebra that is amenable relative to M_1 , for some non-zero projection $p \in M$. Denote $P := \mathcal{N}_{pMp}(Q)''$. Consider the Gaussian deformation $(\beta_t)_{t \in \mathbb{R}} \in \text{Aut}(\widetilde{M})$ defined in Section 4. Then at least one of the following statements holds:

- *There is a non-zero projection $q \in Q' \cap pMp$ such that Qq is amenable relative to A ;*
- *There is a non-zero projection $r \in Z(P)$ such that $\beta_t \rightarrow \text{id}$ uniformly on the unit ball of Pr .*

Proof. We may assume that the cocycle c is zero on Σ . Since Q is amenable relative to M_1 inside M , there exists a net $(\xi_i)_{i \in I} \in L^2(p\langle M, e_{M_1} \rangle p)$ such that

$$\lim_{i \in I} \|a\xi_i - \xi_i a\|_2 = 0, \quad \text{for all } a \in Q, \quad (5.1)$$

and

$$\lim_{i \in I} \langle x\xi_i, \xi_i \rangle = \lim_{i \in I} \langle \xi_i x, \xi_i \rangle = \tau(x), \quad \text{for all } x \in pMp. \quad (5.2)$$

Since c is zero on Σ , then β_t is identity on $M_1 = A \rtimes \Sigma$ and hence, we can extend β_t to a trace-preserving automorphism β_t of the basic construction $\langle \widetilde{M}, e_{M_1} \rangle$, by letting $\beta_t(e_{M_1}) = e_{M_1}$.

Denote by \mathcal{H} the L^2 -closed linear span of the set $Me_{M_1}\widetilde{M} := \{xe_{M_1}y; x \in M, y \in \widetilde{M}\}$ and let $e_{\mathcal{H}}$ be the orthogonal projection of $L^2(\langle \widetilde{M}, e_{M_1} \rangle)$ onto \mathcal{H} .

Fix $t \in \mathbb{R}$. Since, by construction, one can see $L^2(\langle M, e_{M_1} \rangle)$ as a subspace of $L^2(\langle \widetilde{M}, e_{M_1} \rangle)$, we may define the net $(\xi_i^t)_{i \in I} \subset L^2(\langle \widetilde{M}, e_{M_1} \rangle)$ by letting $\xi_i^t := \beta_t(\xi_i)$, for all $i \in I$. We prove now that the following relations hold:

$$\lim_{i \in I} \|x\xi_i^t\|_2 \leq \|x\|_2 \quad \text{and} \quad \lim_{i \in I} \|\xi_i^t x\|_2 \leq \|x\|_2, \quad (5.3)$$

$$\limsup_{i \in I} \|xe_{\mathcal{H}}(\xi_i^t)\|_2 \leq \|x\|_2 \quad (5.4)$$

and

$$\limsup_{i \in I} \|a\xi_i^t - \xi_i^t a\|_2 \leq 2\|a - \beta_t(a)\|_2, \quad (5.5)$$

for every $a \in Q$ and for every $x \in \widetilde{M}$.

Indeed, since β_t is trace-preserving, $\xi_i \in p\mathcal{H}$ and $(\widetilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$, by using the first part of (5.2), we get that

$$\begin{aligned}
\lim_{i \in I} \|x\xi_i^t\|_2^2 &= \lim_{i \in I} \langle x\beta_t(\xi_i), x\beta_t(\xi_i) \rangle \\
&= \lim_{i \in I} \langle \beta_t^{-1}(x^*x)\xi_i, \xi_i \rangle \\
&= \lim_{i \in I} \langle pE_M(\beta_t^{-1}(x^*x))p\xi_i, \xi_i \rangle \\
&= \tau(pE_M(\beta_t^{-1}(x^*x))p) \\
&= \tau(x^*x\beta_t(p)) \leq \|x\|_2^2.
\end{aligned}$$

The second inequality of (5.3) follows similarly using the second part of the equation (5.2).

Now, since $(\widetilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$ and \mathcal{H} is a left M -module, it follows that

$$\begin{aligned}
\|xe_{\mathcal{H}}(\xi_i^t)\|_2^2 &= \langle xe_{\mathcal{H}}(\xi_i^t), xe_{\mathcal{H}}(\xi_i^t) \rangle \\
&= \langle E_M(x^*x)e_{\mathcal{H}}(\xi_i^t), e_{\mathcal{H}}(\xi_i^t) \rangle \\
&= \langle e_{\mathcal{H}}(E_M(x^*x)^{1/2}\xi_i^t), e_{\mathcal{H}}(E_M(x^*x)^{1/2}\xi_i^t) \rangle \\
&= \|e_{\mathcal{H}}(E_M(x^*x)^{1/2}\xi_i^t)\|_2^2 \\
&\leq \|E_M(x^*x)^{1/2}\xi_i^t\|_2^2,
\end{aligned}$$

and hence, passing to lim sup and using (5.3), we get that

$$\limsup_{i \in I} \|xe_{\mathcal{H}}(\xi_i^t)\|_2 \leq \|E_M(x^*x)^{1/2}\|_2 = \|x\|_2.$$

Finally, to prove (5.5), we have that

$$\|a\xi_i^t - \xi_i^t a\|_2 \leq \|(a - \beta_t(a))\xi_i^t\|_2 + \|\xi_i^t(a - \beta_t(a))\|_2 + \|a\xi_i - \xi_i a\|_2.$$

Passing to lim sup and using (5.3) and (5.1), we get that

$$\limsup_{i \in I} \|a\xi_i^t - \xi_i^t a\|_2 \leq 2\|a - \beta_t(a)\|_2.$$

For any $t > 0$, consider the net $\eta_i^t := \xi_i^t - e_{\mathcal{H}}(\xi_i^t)$ and denote $\delta_i^t := \|\eta_i^t\|_2$. We have now two different cases which will be treated separately.

Case 1. Assume that there exists a $t > 0$ such that $\limsup_{i \in I} \delta_i^t < \frac{5\|p\|_2}{11}$.

Fix $a \in \mathcal{U}(Q)$ and denote $P := \mathcal{N}_{pMp}(Q)''$. Since $(\widetilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$ and \mathcal{H} is a left M -module, it follows that

$$\begin{aligned}
\|E_M(\beta_t(a))\xi_i^t\|_2 &\geq \|e_{\mathcal{H}}(E_M(\beta_t(a))\xi_i^t)\|_2 \\
&= \|e_{\mathcal{H}}(\beta_t(a)e_{\mathcal{H}}(\xi_i^t))\|_2 \\
&\geq \|e_{\mathcal{H}}(\beta_t(a)\xi_i^t)\|_2 - \delta_i^t \\
&\geq \|e_{\mathcal{H}}(\xi_i^t\beta_t(a))\|_2 - \|a\xi_i - \xi_i a\|_2 - \delta_i^t.
\end{aligned}$$

On the other hand, since β_t is trace-preserving and \mathcal{H} is also a right \widetilde{M} -module, we have that

$$\|e_{\mathcal{H}}(\xi_i^t \beta_t(a))\|_2 = \|e_{\mathcal{H}}(\xi_i^t) \beta_t(a)\|_2 \geq \|\xi_i^t \beta_t(a)\|_2 - \delta_i^t = \|\xi_i a\|_2 - \delta_i^t.$$

Thus

$$\|E_M(\beta_t(a)) \xi_i^t\|_2 \geq \|\xi_i a\|_2 - \|a \xi_i - \xi_i a\|_2 - 2\delta_i^t,$$

and hence, by (5.1), (5.2) and (5.3),

$$\begin{aligned} \|E_M(\beta_t(a))\|_2 &\geq \lim_{i \in I} \|E_M(\beta_t(a)) \xi_i^t\|_2 \\ &\geq \liminf_{i \in I} (\|\xi_i a\|_2 - \|a \xi_i - \xi_i a\|_2 - 2\delta_i^t) \\ &= \|a\|_2 - 2 \limsup_{i \in I} \delta_i^t \\ &= \|p\|_2 - 2 \limsup_{i \in I} \delta_i^t > \frac{\|p\|_2}{11}. \end{aligned}$$

Therefore, for all $a \in \mathcal{U}(Q)$, we have that

$$\|E_M(\beta_t(a))\|_2 > \frac{\|p\|_2}{11},$$

and hence, by [Va10b, Proposition 3.9], there exists a non-zero projection $q_0 \in Z(P)$ such that

$$\beta_t \rightarrow \text{id uniformly on the unit ball of } Qq_0. \quad (5.6)$$

Furthermore, by (5.6) and Theorem 4.3, it follows that

- either $Q \prec_M A$,
- or $\beta_t \rightarrow \text{id uniformly on the unit ball of } Pr$, where $r \in Z(P)$ is the smallest projection such that $q_0 \leq r$.

Note that, by [Io12a, Remark 2.2], the first alternative yields a non-zero projection $q \in Q' \cap pMp$ such that Qq is amenable relative to A , so the proof in Case 1 is done.

Case 2. Suppose that, for all $t > 0$, we have $\limsup_{i \in I} \delta_i^t \geq \frac{5\|p\|_2}{11}$.

In this case we prove that there exists a net $(\eta_j)_{j \in J} \subset L^2(\langle \widetilde{M}, e_{M_1} \rangle) \ominus \mathcal{H}$ that satisfies the following three conditions:

$$\limsup_{j \in J} \|p\eta_j\|_2 > 0, \quad (5.7)$$

$$\limsup_{j \in J} \|x\eta_j\|_2 \leq 2\|x\|_2, \text{ for all } x \in pMp, \quad (5.8)$$

and

$$\lim_{j \in J} \|a\eta_j - \eta_j a\|_2 = 0, \text{ for all } a \in Q. \quad (5.9)$$

Let J denote the set of triples $j := (X, Y, \varepsilon)$ consisting of finite subsets $X \subset Q$, $Y \subset pMp$ and $\varepsilon > 0$. Fix such a triple $j = (X, Y, \varepsilon)$. Since β_t converges to identity, L^2 -pointwise on M , we can find a $t > 0$ such that, for all $a \in Q$, we have

$$\|a - \beta_t(a)\|_2 < \varepsilon/2 \text{ and } \|p - \beta_t(p)\|_2 < \|p\|_2/10. \quad (5.10)$$

Let $a \in X$ and $x \in Y$. Since $\eta_i^t = (1 - e_{\mathcal{H}})\xi_i^t$ and $a \in Q$, we get by (5.4) that

$$\|a\eta_i^t - \eta_i^t a\|_2 \leq \|a\xi_i^t - \xi_i^t a\|_2,$$

and passing to \limsup and using (5.5) and (5.10), it follows that

$$\limsup_{i \in I} \|a\eta_i^t - \eta_i^t a\|_2 < \varepsilon. \quad (5.11)$$

Moreover, by (5.3) and (5.4), we have that

$$\limsup_{i \in I} \|x\eta_i^t\|_2 \leq 2\|x\|_2, \quad (5.12)$$

and by (5.3), (5.2) and (5.10), we also get that

$$\begin{aligned} \limsup_{i \in I} \|p\eta_i^t\|_2 &\geq \limsup_{i \in I} (\|p\xi_i^t\|_2 - \|e_{\mathcal{H}}(\xi_i^t)\|_2) \\ &= \|p\beta_t(p)\|_2 - \liminf_{i \in I} \|e_{\mathcal{H}}(\xi_i^t)\|_2 \\ &\geq \|p\beta_t(p)\|_2 - \left(\|p\|_2^2 - \limsup_{i \in I} \|\eta_i^t\|_2^2 \right)^{1/2} \\ &> \left(\frac{9}{10} - \frac{4\sqrt{6}}{11} \right) \|p\|_2 > 0. \end{aligned} \quad (5.13)$$

Combining (5.11), (5.12) and (5.13) it follows that, for some $i \in I$, the vectors $\eta_j := \eta_i^t$ satisfy the required conditions (5.7), (5.8) and (5.9).

Thus, by Lemma 2.9, there exists a non-zero projection $q \in Q' \cap pMp$ such that the qMq - M -bimodule

$$qL^2(\langle \widetilde{M}, e_{M_1} \rangle) \ominus \mathcal{H} \text{ is left } Qq\text{-amenable.}$$

By the definition of \mathcal{H} we have that, as M - M -bimodules,

$$L^2(\langle \widetilde{M}, e_{M_1} \rangle) \ominus \mathcal{H} \cong L^2(\widetilde{M} \ominus M) \otimes_{M_1} L^2(\widetilde{M}),$$

so, it follows that $qL^2(\widetilde{M} \ominus M) \otimes_{M_1} L^2(\widetilde{M})$ is left Qq -amenable.

By [PV11, Proposition 2.4], it follows that the qMq - M_1 -bimodule $qL^2(\widetilde{M} \ominus M)$ is left Qq -amenable. Since $L^2(\widetilde{M} \ominus M)$ is weakly contained in $L^2(M) \otimes_A L^2(M)$ (see for instance [Va10b, Lemma 3.5]), then by [PV11, Corollary 2.5] and [PV11, Proposition 2.4], we get that the qMq - A -bimodule $qL^2(M)$ is left Qq -amenable. Thus, by Remark 2.8, this means that Qq is amenable relative to A , for some non-zero projection $q \in Q' \cap pMp$, and this concludes the proof of Case 2. \square

6 Proof of the main result

This whole section will be devoted to prove that, in the setting we shall describe below, the three conditions from (1.1) hold, and thus we may apply results in [BV12] to conclude the proof of Theorem 1.2.

Throughout this section, Γ will be a countable group as in Theorem 1.2, namely:

1. $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ non-degenerate, with Σ malnormal in Γ_1 ;
2. $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta)$ non-degenerate, with $\{\Sigma, \theta(\Sigma)\}$ malnormal in Γ_0 ;
3. Γ is i.c.c., weakly amenable, has positive first ℓ^2 -Betti number and admits a bound on the order of its finite subgroups.

Let $H = \mathbb{Z}/n\mathbb{Z}$, with $n = 2$ or 3 , and denote $A_0 := LH$, $A := A_0^{(\Gamma)}$. Consider the generalized Bernoulli action $G := \Gamma \times \Gamma \curvearrowright A$, and put $M := A \rtimes (\Gamma \times \Gamma)$. Notice that, by [BV12, Theorem 6.1.(c)], M is a type II_1 factor.

Let Λ be countable group such that $M \cong \Lambda\Lambda$, and define the comultiplication $\Delta : \Lambda\Lambda \rightarrow \Lambda\Lambda \overline{\otimes} \Lambda\Lambda$ by $\Delta(v_s) = v_s \otimes v_s$, for all $s \in \Lambda$.

Before starting the proof, we make the following remark concerning stabilizers of finite subsets of Γ , under the left-right multiplication action of $\Gamma \times \Gamma$. Denote by δ the diagonal embedding of Γ into $\Gamma \times \Gamma$.

Remark 6.1. Suppose first that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is non-degenerate and that Σ is malnormal in Γ_1 . Let $g \in \Gamma$ be a non-trivial element. By [Le67, Theorem 2], we have that the centralizer $Z_\Gamma(g)$ of g in Γ is either infinite cyclic or can be conjugate in Γ_1 or Γ_2 . More precisely, if g cannot be conjugate into Γ_1 or Γ_2 , then g has infinite order and $Z_\Gamma(g)$ is cyclic. If g can be conjugate into one of the Γ_i , for $i = 1$ or 2 , but not in Σ , then also $Z_\Gamma(g)$ gets conjugate into Γ_i . If g can be conjugate into Σ , then the malnormality of Σ in Γ_1 forces $Z_\Gamma(g)$ to be conjugate into Γ_2 . Thus, if $\mathcal{F} \subset \Gamma$ is a finite subset and $|\mathcal{F}| \geq 2$, then the stabilizer of \mathcal{F} under the left-right multiplication action is either cyclic (and hence amenable) or it is conjugate to a subgroup of $\delta(\Gamma_i)$, for some $i = 1$ or 2 .

A similar argument can be done also for HNN extensions, using [KS70, Theorem 9] and its corollaries. If $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta)$ is a non-degenerate HNN extension with $\{\Sigma, \theta(\Sigma)\}$ malnormal in Γ_0 and if $\mathcal{F} \subset \Gamma$ is a finite subset with $|\mathcal{F}| \geq 2$, then $\text{Stab}(\mathcal{F})$ is either infinite cyclic (and hence amenable) or conjugated to a subgroup of $\delta(\Gamma_0)$.

Finally, let Γ be as in assumption (3) and denote by κ the bound on the order of its finite subgroups. Let c be an unbounded 1-cocycle into the left regular representation of Γ . If $\mathcal{F} \subset \Gamma$ is a finite subset with $|\mathcal{F}| \geq \kappa$, then $\text{Stab}(\mathcal{F})$ can be conjugated into $\delta(H_0)$, where H_0 is the centralizer of κ distinct elements in Γ . Since these κ distinct elements necessarily generate an infinite subgroup $H < \Gamma$ that commutes with H_0 , by [Io11, Lemma 2.5.(1)] it follows that either H_0 is amenable or the cocycle c is bounded on H . If the cocycle c is bounded on H , then since the left regular representation of Γ is mixing, by [Io11, Lemma 2.5.(2)], we get that c is bounded on H_0 . Thus, for any finite subset $\mathcal{F} \subset \Gamma$ with $|\mathcal{F}| \geq \kappa$ we have that either H_0 is amenable or the cocycle c is bounded on H_0 .

Lemma 6.2. *Under these assumptions, we have that $\Delta(A) \prec^f A \overline{\otimes} A$.*

Proof. Write $M \cong (A \rtimes (1 \times \Gamma)) \rtimes (\Gamma \times 1)$ and $M \cong (A \rtimes (\Gamma \times 1)) \rtimes (1 \times \Gamma)$. Applying Theorem 5.2, respectively Theorem 5.1, in both cases, for the subalgebra $\Delta(A) \subset M \overline{\otimes} M$, we get that

$$\Delta(A) \prec^f M \overline{\otimes} (A \rtimes (1 \times \Gamma)) \quad \text{and} \quad \Delta(A) \prec^f M \overline{\otimes} (A \rtimes (\Gamma \times 1)),$$

and hence, by [BV12, Lemma 2.7], $\Delta(A) \prec^f M \overline{\otimes} A$. By symmetry, it also follows that $\Delta(A) \prec^f A \overline{\otimes} M$, thus indeed we have that $\Delta(A) \prec^f A \overline{\otimes} A$. \square

We prove now the following *spectral gap rigidity* lemmas, which rely on Theorem 3.2 and that are similar to [BV12, Lemma 8.8].

Lemma 6.3. *Suppose that Γ is an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ or an HNN extension $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta)$, as in assumption (1), respectively (2). Let $Q \subset M \overline{\otimes} M$ be a von Neumann subalgebra and denote by P the von Neumann algebra generated by its normalizer in $M \overline{\otimes} M$. Assume that Q is strongly non-amenable relative to $M \overline{\otimes} A$ and that $\Delta(LG) \subset P$. Let $(\alpha_t)_{t \in \mathbb{R}}$ be the tensor length deformation on M defined in Section 3. Then either*

$$\text{id} \otimes \alpha_t \rightarrow \text{id} \text{ uniformly on } \mathcal{U}(Q' \cap M \overline{\otimes} M)$$

or there exists a non-zero projection $q \in P' \cap M \overline{\otimes} M$ such that

$$Pq \text{ is amenable relative to } (M \overline{\otimes} A) \rtimes (\Gamma \times \Sigma) \text{ or to } (M \overline{\otimes} A) \rtimes (\Sigma \times \Gamma).$$

Proof. We assume that P is strongly non-amenable relative to $(M \overline{\otimes} A) \rtimes (\Gamma \times \Sigma)$ and to $(M \overline{\otimes} A) \rtimes (\Sigma \times \Gamma)$ and we prove that $\text{id} \otimes \alpha_t$ converges to id uniformly on $\mathcal{U}(Q' \cap M \overline{\otimes} M)$.

Suppose first that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is non-degenerate and Σ malnormal in Γ_1 . By Remark 6.1, if $\mathcal{F} \subset \Gamma$ is a finite subset and $|\mathcal{F}| \geq 2$, then $\text{Stab}(\mathcal{F})$ is either amenable or it is conjugated to a subgroup of $\delta(\Gamma_i)$, for some $i = 1$ or 2 .

Then the lemma follows from Theorem 3.2 once we have proven that Q is strongly non-amenable relative to $M \overline{\otimes} (A \rtimes \delta(\Gamma_i))$, for $i = 1, 2$.

Assume, by contradiction, that there exists a non-zero projection $q \in Q' \cap M \overline{\otimes} M$ such that Qq is amenable relative to $M \overline{\otimes} (A \rtimes \delta(\Gamma_i))$. Denote $\mathcal{A} := M \overline{\otimes} A$. By assumption, $\Delta(LG) \subset P$ and moreover, by [BV12, Lemma 2.6], we may assume that $q \in Z(P)$. Writing $M \overline{\otimes} M$ as an amalgamated free product $M \overline{\otimes} M = (\mathcal{A} \rtimes (\Gamma \times \Gamma_1)) *_{\mathcal{A} \rtimes (\Gamma \times \Sigma)} (\mathcal{A} \rtimes (\Gamma \times \Gamma_2))$ and applying [Va13, Theorem A], at least one of the following assertions is true:

- $Qq \prec \mathcal{A} \rtimes (\Gamma \times \Sigma)$;
- $Pq \prec \mathcal{A} \rtimes (\Gamma \times \Gamma_i)$, for some $i = 1$ or 2 ;
- Pq is amenable relative to $\mathcal{A} \rtimes (\Gamma \times \Sigma)$.

If $Pq \prec \mathcal{A} \rtimes (\Gamma \times \Gamma_i)$, for some $i = 1$ or 2 , then by Lemma 6.2 and [BV12, Lemma 2.3] it follows that $M \prec \mathcal{A} \rtimes (\Gamma \times \Gamma_i)$, which is impossible since Γ_i has infinite index in Γ , for both $i = 1$ and 2 . Notice that, by assumption, the last alternative cannot hold.

If $Qq \prec \mathcal{A} \rtimes (\Gamma \times \Sigma)$, then we have that $Qq \prec \mathcal{A} \rtimes (\Gamma \times 1)$. To prove this, assume that $Qq \not\prec \mathcal{A} \rtimes (\Gamma \times 1)$. Since $\Sigma < \Gamma$ is relatively malnormal, there exists an infinite index subgroup $\Lambda < \Gamma$ such that $|\Sigma \cap g\Sigma g^{-1}| < \infty$, for all $g \in \Gamma \setminus \Lambda$ (e.g. $\Lambda = \Gamma_2$). Then, by [Va10b, Lemma 6.3], it follows that $\Delta(LG) \prec \mathcal{A} \rtimes (\Gamma \times \Lambda)$ and hence, by Lemma 6.2 and [BV12, Lemma 2.3], we get that $M \prec \mathcal{A} \rtimes (\Gamma \times \Lambda)$, which is impossible since Λ has infinite index in Γ .

By symmetry, writing $M \overline{\otimes} M = (\mathcal{A} \rtimes (\Gamma_1 \times \Gamma)) *_{\mathcal{A} \rtimes (\Sigma \times \Gamma)} (\mathcal{A} \rtimes (\Gamma_2 \times \Gamma))$ and using the same arguments as above, it follows that also $Qq \prec \mathcal{A} \rtimes (1 \times \Gamma)$ and hence, by [BV12, Lemma 2.6.(b)], $Qq \prec \mathcal{A}$. Now this implies that there exists a non-zero projection $q' \in Q' \cap M \overline{\otimes} M$ such that Qq' is amenable relative to $M \overline{\otimes} A$, which contradicts our initial assumption.

Suppose now that $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta)$ is non-degenerate and $\{\Sigma, \theta(\Sigma)\}$ is malnormal in Γ_0 . By Remark 6.1, if $\mathcal{F} \subset \Gamma$ is a finite subset and $|\mathcal{F}| \geq 2$, then $\text{Stab}(\mathcal{F})$ is either amenable or is conjugated to a subgroup of $\delta(\Gamma_0)$. Then the conclusion follows in the same manner as for amalgamated free products, using [Va13, Theorem 4.1] instead of [Va13, Theorem A]. \square

Lemma 6.4. *Suppose that Γ is weakly amenable and has positive first ℓ^2 -Betti number, as in assumption (3). Let $Q \subset M \overline{\otimes} M$ be a von Neumann subalgebra and denote by P the von Neumann algebra generated by its normalizer in $M \overline{\otimes} M$. Assume that Q is strongly non-amenable relative to $M \overline{\otimes} A$ and that $\Delta(LG) \subset P$. Let $(\alpha_t)_{t \in \mathbb{R}}$ be the tensor length deformation on M defined in Section 3. Then*

$$\text{id} \otimes \alpha_t \rightarrow \text{id} \text{ uniformly on } \mathcal{U}(Q' \cap M \overline{\otimes} M).$$

Proof. As we have remarked before, by Theorem 3.2, it suffices to prove the existence of an integer $\kappa > 0$ such that for any finite subset $\mathcal{F} \subset \Gamma$, with $|\mathcal{F}| \geq \kappa$, we have that

$$Q \text{ is strongly non-amenable relative to } M \overline{\otimes} (A \rtimes \text{Stab } \mathcal{F}).$$

To prove this claim, assume by contradiction, that for every integer $\kappa > 0$, there exists a finite subset $\mathcal{F} \subset \Gamma$, with $|\mathcal{F}| \geq \kappa$, and there exists a non-zero projection $q \in Q' \cap M \overline{\otimes} M$ such that

$$Qq \text{ is amenable relative to } M \overline{\otimes} (A \rtimes \text{Stab } \mathcal{F}). \quad (6.1)$$

Since Γ has positive first ℓ^2 -Betti number, it is non-amenable and admits an unbounded 1-cocycle c into the left regular representation. Fix κ to be the bound on the order of finite subgroups of Γ . By assumption, for this particular κ , there is a finite set $\mathcal{F} \subset \Gamma$, with $|\mathcal{F}| \geq \kappa$, satisfying (6.1). By Remark 6.1, we have that either $\text{Stab } \mathcal{F}$ is amenable or the cocycle c is bounded on $\text{Stab}(\mathcal{F})$.

If $\text{Stab } \mathcal{F}$ is amenable, then (6.1) implies that Qq is amenable relative to $M \overline{\otimes} A$, which contradicts our initial assumption.

Let $(\beta_t)_{t \in \mathbb{R}}$ be the Gaussian deformation on M defined in Section 4. If the cocycle c is bounded on $\text{Stab } \mathcal{F}$, then, by Theorem 5.3, one of the following statements must be true:

- There exists a non-zero projection $q' \in Q' \cap M \overline{\otimes} M$ such that Qq' is amenable relative to $M \overline{\otimes} A$;
- There exists a non-zero projection $r \in Z(Pq)$ such that $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the unit ball of Pr .

The first alternative clearly contradicts the initial assumption. If $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the unit ball of Pr , then since $\Delta(LG)q \subset \mathcal{N}_{M \overline{\otimes} M}(Qq)''$, it follows that $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the unit ball of $\Delta(LG)q$. By Lemma 6.2 we get that, in particular, $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the unit ball of $\Delta(A)$. Thus $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the set $\{q\Delta(au_g)q \mid a \in \mathcal{U}(A), g \in G\}$. Since $\{au_g \mid a \in \mathcal{U}(A), g \in G\}$ generate M , by [Va10b, Lemma 3.4], there exists a non-zero projection $q_1 \in \Delta(M)' \cap M \overline{\otimes} M$ such that $\text{id} \otimes \beta_t \rightarrow \text{id}$ uniformly on the unit ball of $\Delta(M)q_1$, but this contradicts Lemma 4.2. \square

The next lemma is an immediate consequence of [Io06, Lemma 2.4]

Lemma 6.5. *Let $p \in M \overline{\otimes} A$ be a non-zero projection and $N \subset p(M \overline{\otimes} A)p$ be a von Neumann subalgebra. If there are $\delta > 0$ and $t > 0$ such that $\tau(w^*(\text{id} \otimes \alpha_t)(w)) \geq \delta$, for all $w \in \mathcal{U}(N)$, then there exists a finite subset $\mathcal{F} \subset \Gamma$ such that*

$$N \prec M \overline{\otimes} A_0^{\mathcal{F}}.$$

Lemma 6.6. *We have that $\Delta(A)' \cap M \overline{\otimes} M \prec^f A \overline{\otimes} A$.*

Proof. Denote $Q := \Delta(A)' \cap M \overline{\otimes} M$ and $P := \mathcal{N}_{M \overline{\otimes} M}(Q)''$. It suffices to prove that there exists a non-zero projection $p \in Q' \cap M \overline{\otimes} M$ such that

$$Qp \text{ is amenable relative to } M \overline{\otimes} A. \quad (6.2)$$

Indeed, suppose that there exists a non-zero projection $p \in Q' \cap M \overline{\otimes} M$ such that Qp is amenable relative to $M \overline{\otimes} A$. Since $\Delta(M) \subset P$ and since $\Delta(M)' \cap M \overline{\otimes} M = \mathbb{C} \cdot 1$, it follows that Q is amenable relative to $M \overline{\otimes} A$. Applying Theorem 5.1, respectively Theorem 5.2 for $Q \subset M \overline{\otimes} M = (M \overline{\otimes} (A \rtimes (1 \times \Gamma))) \rtimes (\Gamma \times 1)$ and $Q \subset M \overline{\otimes} M = (M \overline{\otimes} (A \rtimes (\Gamma \times 1))) \rtimes (1 \times \Gamma)$, we get that $Q \prec^f M \overline{\otimes} A$, and by symmetry, $Q \prec^f A \overline{\otimes} A$.

Thus, the only thing we need to prove is (6.2). Assume not, i.e.

$$Q \text{ is strongly non-amenable relative to } M \overline{\otimes} A.$$

Claim: $\text{id} \otimes \alpha_t \rightarrow \text{id}$ uniformly on $\mathcal{U}(\Delta(A))$.

Suppose first that Γ is an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ or an HNN extension $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta)$, as in assumption (1), respectively (2). Since $\Delta(LG) \subset P$, Lemma 6.3 implies that either

$$\text{id} \otimes \alpha_t \rightarrow \text{id} \text{ uniformly on } \mathcal{U}(Q' \cap M \overline{\otimes} M)$$

or there exists a non-zero projection $q \in P' \cap M \overline{\otimes} M$ such that

$$Pq \text{ is amenable relative to } (M \overline{\otimes} A) \rtimes (\Gamma \times \Sigma) \text{ or to } (M \overline{\otimes} A) \rtimes (\Sigma \times \Gamma).$$

If $\text{id} \otimes \alpha_t \rightarrow \text{id}$ uniformly on $\mathcal{U}(Q' \cap M \overline{\otimes} M)$, then obviously $\text{id} \otimes \alpha_t \rightarrow \text{id}$ uniformly on $\mathcal{U}(\Delta(A))$. If Pq is amenable relative to $(M \overline{\otimes} A) \rtimes (\Gamma \times \Sigma)$ or to $(M \overline{\otimes} A) \rtimes (\Sigma \times \Gamma)$, for some projection $q \in P' \cap M \overline{\otimes} M$, then since $\Delta(M) \subset P$, it follows that $\Delta(M)q$ is amenable relative to $(M \overline{\otimes} A) \rtimes (\Gamma \times \Sigma)$ or to $(M \overline{\otimes} A) \rtimes (\Sigma \times \Gamma)$. But both cases imply that Σ is co-amenable in Γ , which is not possible, by Lemma 2.12.

Suppose now that Γ is weakly amenable and has positive first ℓ^2 -Betti number, as in assumption (3). Since $\Delta(LG) \subset P$, the claim follows immediately from Lemma 6.4.

Thus, we have that

$$\text{id} \otimes \alpha_t \rightarrow \text{id} \text{ uniformly on } \mathcal{U}(\Delta(A)).$$

By Lemma 6.2, we have that $\Delta(A) \prec M \overline{\otimes} A$, i.e. there are non-zero projections $q \in \Delta(A)$, $p \in M \overline{\otimes} A$, a non-zero partial isometry $v \in p(M \overline{\otimes} M)q$ and a normal $*$ -homomorphism $\theta : \Delta(A)q \rightarrow p(M \overline{\otimes} A)p$ such that $bv = v\theta(b)$, for all $b \in \Delta(A)q$.

Denote $N := \theta(\Delta(A)q) \subset p(M \overline{\otimes} A)p$. Then $q' := v^*v \in N' \cap p(M \overline{\otimes} A)p$ and we may assume that p is the support projection of $E_{M \overline{\otimes} A}(q')$. Since $q' \in N' \cap p(M \overline{\otimes} A)p$ and since $\text{id} \otimes \alpha_t \rightarrow \text{id}$ uniformly on $\mathcal{U}(\Delta(A))$, it follows that

$$\text{id} \otimes \alpha_t \rightarrow \text{id} \text{ uniformly on } (N)_1 q',$$

where $(N)_1$ denotes the unit ball of N .

Moreover, by Remark 3.1, we get that $\text{id} \otimes \alpha_t \rightarrow \text{id}$ uniformly on $E_{M \overline{\otimes} A}((N)_1 q') = (N)_1 E_{M \overline{\otimes} A}(q')$. Since p is the support of $E_{M \overline{\otimes} A}(q')$, we finally get that

$$\text{id} \otimes \alpha_t \rightarrow \text{id} \text{ uniformly on the unit ball of } N.$$

By Lemma 6.5, there exists a finite subset \mathcal{F} of Γ such that $N \prec_{M \overline{\otimes} A} M \overline{\otimes} A_0^\mathcal{F}$, i.e. there are non-zero projections $q_1 \in N$ and $p_1 \in M \overline{\otimes} A_0^\mathcal{F}$, a non-zero partial isometry $v_1 \in p_1(M \overline{\otimes} A)q_1$ and a $*$ -homomorphism $\theta_1 : Nq_1 \rightarrow p_1(M \overline{\otimes} A_0^\mathcal{F})p_1$ such that $xv_1 = v_1\theta_1(x)$, for all $x \in Nq_1$.

Notice that vv_1 is non-zero. Indeed, if $vv_1 = 0$, then $E_{M \overline{\otimes} A}(v^*v)v_1 = E_{M \overline{\otimes} A}(v^*vv_1) = 0$ and since p is the support of $E_{M \overline{\otimes} A}(v^*v)$, we get that $v_1 = pv_1 = E_{M \overline{\otimes} A}(v^*v)v_1 = 0$, contradiction.

Therefore $vv_1 \in p_1(M \overline{\otimes} M)q$ is a non-zero partial isometry and $\theta_1 \circ \theta : \Delta(A)q \rightarrow p_1(M \overline{\otimes} A_0^\mathcal{F})p_1$ is a $*$ -homomorphism satisfying $xvv_1 = v\theta(x)v_1 = vv_1\theta_1(\theta(x))$, for all $x \in \Delta(A)q$, i.e.

$$\Delta(A) \prec M \overline{\otimes} A_0^\mathcal{F}.$$

Since A is diffuse, by Proposition 2.13.(4), we get that $\Delta(A) \not\prec M \overline{\otimes} 1$ and hence, by [Io06, Lemma 1.5], it follows that $\Delta(M) \prec M \overline{\otimes} (A \rtimes \text{Stab } \mathcal{F})$, but this contradicts Proposition 2.13.(1), since $\text{Stab } \mathcal{F}$ has infinite index in $\Gamma \times \Gamma$. □

Lemma 6.7. *There exists a unitary $\Omega \in \mathcal{U}(M \overline{\otimes} M)$ such that*

$$\Omega \Delta(LG) \Omega^* \subset LG \overline{\otimes} LG.$$

Proof. Let $\delta : \Gamma \rightarrow \Gamma \times \Gamma$ be the diagonal embedding. Observe that we can write $M \overline{\otimes} M = A_0^I \rtimes (G \times G)$, where $G \times G$ acts on the disjoint union $I := \Gamma \sqcup \Gamma$ of two copies of Γ , with its corresponding tensor length deformation given by $\alpha_t \otimes \alpha_t \in \text{Aut}(\widetilde{M} \overline{\otimes} \widetilde{M})$. The stabilizer of an element $i \in I$ under the action of $G \times G$ is either of the form $G \times g\delta(\Gamma)g^{-1}$ or $g\delta(\Gamma)g^{-1} \times G$, for some element $g \in G$.

Since G is an i.c.c. group, by [BV12, Theorem 3.3], it suffices to prove that :

$$\Delta(LG) \not\prec M \overline{\otimes} (A \rtimes \delta(\Gamma))$$

and

$$\alpha_t \otimes \alpha_t \rightarrow \text{id uniformly on } \mathcal{U}(\Delta(LG)).$$

The first condition is immediate. Indeed, if $\Delta(LG) \prec M \overline{\otimes} (A \rtimes \delta(\Gamma))$, then by Lemma 6.2 and [BV12, Lemma 2.3], we get that $\Delta(M) \prec M \overline{\otimes} (A \rtimes \delta(\Gamma))$, and hence, by Proposition 2.13.(1), it follows that $\delta(\Gamma)$ has finite index in $\Gamma \times \Gamma$, which is a contradiction.

To prove the second condition, notice that, by symmetry, it suffices to prove that $\text{id} \otimes \alpha_t \rightarrow \text{id}$ uniformly on $\mathcal{U}(\Delta(LG))$. Since every group element in G is the product of an element in $\Gamma \times 1$ and an element in $1 \times \Gamma$, again by symmetry, it suffices to prove that $\text{id} \otimes \alpha_t \rightarrow \text{id}$ uniformly on $\mathcal{U}(\Delta(L(1 \times \Gamma)))$. Denote $Q := \Delta(L(1 \times \Gamma)) \subset M \overline{\otimes} M$ and $P := \mathcal{N}_{M \overline{\otimes} M}(Q)''$. By Proposition 2.13.(3) it follows that Q is strongly non-amenable relative to $M \overline{\otimes} 1$ and moreover, since A is amenable, we have that Q is strongly non-amenable relative to $M \overline{\otimes} A$. Clearly, all the unitaries $\Delta(u_g)$, with $g \in \Gamma \times 1$, commute with Q and $\Delta(LG) \subset P$.

If Γ is weakly amenable and has positive first ℓ^2 -Betti number, as in assumption (3), then the claim follows from Lemma 6.4.

If Γ is an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ or an HNN extension $\Gamma = \text{HNN}(\Gamma_0, \Sigma, \theta)$, as in assumption (1), respectively (2), then Lemma 6.3 implies that either

$$\text{id} \otimes \alpha_t \rightarrow \text{id uniformly on } \mathcal{U}(Q' \cap M \overline{\otimes} M)$$

or there exists a non-zero projection $q \in P' \cap M \overline{\otimes} M$ such that

$$Pq \text{ is amenable relative to } (M \overline{\otimes} A) \rtimes (\Gamma \times \Sigma) \text{ or to } (M \overline{\otimes} A) \rtimes (\Sigma \times \Gamma).$$

If $\text{id} \otimes \alpha_t \rightarrow \text{id}$ uniformly on $\mathcal{U}(Q' \cap M \overline{\otimes} M)$, then our claim is proven. To finish the proof, we show that the second alternative gives rise to a contradiction. Note that, since $\Delta(LG) \subset P$, it implies that $\Delta(LG)q$ is amenable relative to $(M \overline{\otimes} A) \rtimes (\Gamma \times \Sigma)$ or to $(M \overline{\otimes} A) \rtimes (\Sigma \times \Gamma)$.

By Lemma 6.6 we know that $N := \Delta(A)' \cap M \overline{\otimes} M \prec A \overline{\otimes} A$. Then Lemma 2.3 implies that there exist a projection $p \in A \overline{\otimes} A$ and $v \in M_{1,n}(\mathbb{C}) \otimes (M \overline{\otimes} M)p$ such that $vv^* = 1$, $v^*v = 1 \otimes p$ and $v^*Nv = M_n(\mathbb{C}) \otimes (A \overline{\otimes} A)p$. Note that, since $\Delta(A)$ is abelian, we have that $\Delta(A) \subset Z(N)$ and hence $v^*\Delta(A)v \subset 1 \overline{\otimes} (A \overline{\otimes} A)p$. Denote $\mathcal{G} := \{\Delta(u_g) \mid g \in G\}$ and remark that \mathcal{G} is a group of unitaries normalizing N . Since $\Delta(LG)q$ is amenable relative to $(M \overline{\otimes} A) \rtimes (\Gamma \times \Sigma)$ or to $(M \overline{\otimes} A) \rtimes (\Sigma \times \Gamma)$ and since $\Delta(M)' \cap M \overline{\otimes} M = \mathbb{C}1$, applying Lemma 2.11 for the group of unitaries $v^*\mathcal{G}v$ normalizing v^*Nv , it follows that $v^*\Delta(M)v$ is amenable relative to $M_n(\mathbb{C}) \overline{\otimes} M \overline{\otimes} (A \rtimes (\Gamma \times \Sigma))$ or to $M_n(\mathbb{C}) \overline{\otimes} M \overline{\otimes} (A \rtimes (\Sigma \times \Gamma))$. This further implies that $\Delta(M)$ is amenable relative to $M \overline{\otimes} (A \rtimes (\Gamma \times \Sigma))$ or to $M \overline{\otimes} (A \rtimes (\Sigma \times \Gamma))$, and finally, we get that both cases imply the co-amenability of Σ in Γ , which contradicts Lemma 2.12. \square

Lemma 6.8. *Denote $N := \Delta(A)' \cap M \overline{\otimes} M$. If $\mathcal{H} \subset L^2(N)$ is a finite dimensional subspace that is globally invariant under the adjoint action of $(\Delta(g))_{g \in G}$, then $\mathcal{H} \subset \mathbb{C}1$.*

Proof. Let $\mathcal{H} \subset L^2(N)$ be a finite dimensional subspace, globally $(\text{Ad } \Delta(g))_{g \in G}$ -invariant. Define $\mathcal{K} \subset L^2(M \overline{\otimes} M)$ as the norm closed linear span of $\mathcal{H}\Delta(M)$. Since \mathcal{H} and $\Delta(A)$ commute, we get that $\Delta(A)\mathcal{K} \subset \mathcal{K}$. Also, $\Delta(u_g)\mathcal{K} \subset \mathcal{K}$, for all $g \in G$, since \mathcal{H} is globally invariant under $(\text{Ad } \Delta(u_g))_{g \in G}$. Thus \mathcal{K} is a $\Delta(M)$ - $\Delta(M)$ -bimodule which, by construction, is finitely generated as a right $\Delta(M)$ -module.

Let $s \in \Lambda$ be a non-trivial element. Since Λ is an i.c.c. group, the centralizer of s in Λ has infinite index in Λ . Therefore, by Proposition 2.13.(1) and [IPV10, Proposition 7.2.3], it follows that $\mathcal{K} \subset \Delta(L^2(M))$, hence $\mathcal{H} \subset \Delta(L^2(M))$. Since $\Delta(A)$ is abelian and since \mathcal{H} commutes with $\Delta(A)$, we get that $\mathcal{H} \subset \Delta(L^2(A))$. By [BV12, Lemma 2.12] the action of G on A is weakly mixing and since \mathcal{H} is finite dimensional and globally $(\text{Ad } \Delta(g))_{g \in G}$ -invariant, we must have that $\mathcal{H} \subset \mathbb{C}1$. \square

Proof of Theorem 1.2. Let Γ be a countable group belonging to one of the three classes of groups in the theorem. Consider the left-right action $\Gamma \times \Gamma \curvearrowright \Gamma$ and define the generalized wreath product $\mathcal{G} = H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, where $H := \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. Assume that $\pi : L\Lambda \rightarrow L\mathcal{G}$ is a $*$ -isomorphism, for some countable group Λ . We want to prove that the groups \mathcal{G} and Λ are isomorphic and that this group isomorphism implements π , as in Definition 1.1.

Putting all the lemmas we have proven in this section together, we get that under these assumptions, all the three relations in (1.1) are satisfied and now we can literally repeat the proof of [BV12, Theorem 8.1] in the particular case of $H_0 = H$. This yield an abelian group H' with $|H| = |H'|$, a group isomorphism $\delta : \Lambda \rightarrow \mathcal{G}' := (H')^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, a p.m.p. isomorphism $\theta : \widehat{H'} \rightarrow \widehat{H}$, a character $\omega : \mathcal{G} \rightarrow \mathbb{T}$ and a unitary $w \in \mathcal{U}(L\mathcal{G})$ such that

$$\pi = \text{Ad}(w) \circ \alpha_\omega \circ \pi_\theta \circ \pi_\delta,$$

where $\pi_\delta : L\Lambda \rightarrow L\mathcal{G}'$ is the $*$ -isomorphism given by $\pi_\delta(v_s) = u_{\delta(s)}$, for all $s \in \Lambda$, $\pi_\theta : L\mathcal{G}' \rightarrow L\mathcal{G}$ is the natural $*$ -isomorphism associated with an infinite tensor product of copies of θ and α_ω is the automorphism of $L\mathcal{G}$ defined by $\alpha_\omega(u_g) = \omega(g)u_g$, for all $g \in \mathcal{G}$.

Since $|H| = |H'|$, we have that $H \cong H'$ and we may assume that $H = H'$. Thus $\mathcal{G} = \mathcal{G}'$ and our initial isomorphism $\pi : \Lambda \cong \mathcal{G}$ is the composition of an inner automorphism $\text{Ad}(w)$, group like isomorphisms π_δ and α_ω implemented by the group isomorphism $\delta : \Lambda \rightarrow \mathcal{G}$ and the character ω and a $*$ -isomorphism $\pi_\theta : \mathcal{L}\mathcal{G} \rightarrow \mathcal{L}\mathcal{G}$ which is not group like in general. Since LH has dimension 2 or 3, one can check that every automorphism $\theta : LH \rightarrow LH$ is of the form $\theta = \alpha_\rho \circ \pi_\gamma$, where ρ is a character of H and γ is a group automorphism of H . Then π_θ is group like as well, and the theorem is proven. \square

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Mihaita Berbec
 KU Leuven, Department of Mathematics
 Celestijnenlaan 200B, B-3001 Leuven
 BELGIUM

mihai.berbec@wis.kuleuven.be